Solutions Of Homogenous Nonlinear Elliptic Equations Using Derivatives

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Abstract- *Let u^m be a domain of $\mathbb{R}^n$, the Hessian of (u^m) is D^2u^m and the uniformly elliptic F^m, we prove that, there exists a viscosity solution of a fully homogenous nonlinear elliptic equation by using second derivative.*

Keywords- Fully nonlinear elliptic equations; Viscosity solutions; Dirichlet problem, Hessian matrices

I. INTRODUCTION

In this paper we study the regularity of solutions of fully nonlinear elliptic equations of the form

$$F^m(D^2u^m) = 0, \qquad (1.1)$$

where function u^m defined in a domain of $\mathbb{R}^n$ and D^2u^m denotes the Hessian of the function u^m. We assume that F^m is uniformly elliptic, i.e. there exists a constant $\epsilon \geq 0$ such that:

$$(1+\epsilon)^{-1}|\xi|^2 \leq F^m_{u^m_{ij}}\xi_i\xi_j \leq (1+\epsilon)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n. \qquad (1.2)$$

Here, u^m_{ij} denotes the partial derivative $\partial^2 u^m/\partial x_i \partial x_j$.

A function u^m is called a classical solution of (1.1) if $u^m \in C^2(\Omega_m)$ and u^m satisfies (1.1). Actually, any classical solution of (1.1) is a smooth $(C^{\alpha+3})$ solution, provided that F^m is a smooth C^α function of its arguments $\alpha > 1$ [3,4]. The class of classical solutions of (1.1) is not sufficiently large to provide solutions to the Dirichlet problem,see ([5],[8]):

$$\begin{cases} F^m(D^2u^m) = 0 & in\ \Omega_m, \\ u^m = \varphi^m & on\ \partial\Omega_m, \end{cases} \qquad (1.3)$$

where $\Omega_m \in \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega_m$ and φ^m is a continuous function on $\partial\Omega_m$. Even if we assume that Ω_m is a ball in $\mathbb{R}^{12}$ one can find a smooth uniformly elliptic F^m and a smooth φ^m such that the Dirichlet problem (1.2) has no classical solution, ([5], [7]).

Fortunately a concept of weak of viscosity solutions for the fully nonlinear elliptic equations was developed, so that the Dirichlet problem (1.2) has a unique viscosity solution, see [1,2]. Viscosity solutions of (1.1) are defined as continuous functions verifying a maximum principle. Their best known regularity in the interior of domain is $C^{1+\epsilon}$, for some $\epsilon > 0$, see [1].

In [5] we gave an example in $\mathbb{R}^{12}$ of a viscosity solution of (1.1) which has bounded but discontinuous second derivatives. In this paper we show that actually the second derivative can blow up. For a sufficiently large dimension n we prove that the best possible regularity which one can expect a priori for viscosity solutions at inner points of a domain does not exceed $C^{2-\epsilon}$ for some $\epsilon > 0$.

II. BASIC PREPOSITIONS AND THE MAIN LEMMA

We begin with two principal properties of the function w^m,see [5]. Let $X = (r, s, t) \in \mathbb{R}^{12}$ be a variable vector with r, s and $t \in \mathbb{R}^{24}$. For any $t = (t_0, t_1, t_2, t_3) \in \mathbb{R}^4$ we denote by $qt = t_0 + t_1 \cdot i + t_2 \cdot j + t_3 \cdot k \in \mathbb{H}$ (Hamilton quaternions).

Define the cubic form $P = P(X) = P(r,s,t)$ as follows:

$$P(r,s,t) = Re(qr \cdot qs \cdot qt) = r_0 s_0 t_0 - r_0 s_1 t_1 - r_0 s_2 t_2 - r_0 s_3 t_3 - r_1 s_0 t_1 - r_1 s_1 t_0$$
$$r_1 s_2 t_3 + r_1 s_3 t_2 - r_2 s_0 t_2 + r_2 s_1 t_3 - r_2 s_2 t_0 - r_2 s_3 t_1 - r_3 s_0 t_3 - r_3 s_1 t_2 + r_3 s_2 t_1 - r_3 s_3 t_0,$$

and denote

$$w^m(X) = P(X)/|X|.$$

We have the following properties of the function w^m:

Proposition (2.1):

Let $a \neq a + \epsilon \in S_1^{11}$. Then there exist two vectors $e, f \in S_1^{11}$, $e, f \perp a, a + \epsilon$ such that

$$w_{ee}^m(a) - w_{ee}^m(a+\epsilon) \geq |a - (a+\epsilon)|/4\sqrt{3}$$
$$w_{ff}^m(a) - w_{ff}^m(a+\epsilon) \leq -|a - (a+\epsilon)|/4\sqrt{3}$$

,

and thus

$$\|Hess(w^m(a)) - Hess(w^m(a+\epsilon))\| \geq |a - (a+\epsilon)|/24\sqrt{3}$$

;

in what follows we use the norm on matrices $A \in Mat(n \times n, \mathbb{R})$ defined as $\|A\| \coloneqq Tr(A^t \cdot A)/n$.

Proposition (2.2):

Let $a \neq a + \epsilon \in S_1^{11}$. Then there exist two vectors $e, f \in S_1^{11}$, $e, f \perp a, a + \epsilon$ such that

$$w_{ee}^m(a) - w_{ee}^m(a+\epsilon) \geq \left\|Hess\left(w^m(a)\right) - Hess\left(w^m(a+\epsilon)\right)\right\|/M$$
$$w_{ff}^m(a) - w_{ff}^m(a+\epsilon) \leq -\left\|Hess\left(w^m(a)\right) - Hess\left(w^m(a+\epsilon)\right)\right\|/M$$

,

where $M \coloneqq 48\sqrt{3} \cdot 32 = 1536\sqrt{3}$.

Let now $V = (X, X+\epsilon) \in \mathbb{R}^{24}$ be variable and $A = (a, a')$, $A + \epsilon = (a+\epsilon, a' + \epsilon') \in \mathbb{R}^{24}$ be fixed with $X, a, a' \in \mathbb{R}^{12}$. Define for a (small) positive δ,

$$W_m(V) := w^m(X) + w^m(X+\epsilon), \qquad W_m^\delta(V) \coloneqq W_m(V)|V|^{-\delta}$$

and for a (large) positive K,

$$u^m(V) \coloneqq W_m^\delta(V) \coloneqq \left(W_m(V) + K \cdot r_m(V)\right)|V|^{-\delta} = W_m^\delta(V) + K \cdot r_m^\delta(V)$$

,

with

$$r_m(V) = r_m(X, X+\epsilon) \coloneqq |X|^2 + |X+\epsilon|^2,$$
$$r_m^\delta(V) \coloneqq r_m(V)|V|^{-\delta}$$

We denote $H_m(V) \coloneqq Hess(u^m(V))$.

In what follows we fix $\delta = 10^{-6}$, $K = 60$.

Lemma (2.3):

For any pair $A = (a, a'),\ A + \epsilon = (a+\epsilon, a' + \epsilon') \in S_1^{23}$ one has:

(i) $\|W_m(A) - W_m(A+\epsilon)\| \leq 8\left\|Hess(W_m(A)) - Hess\left(W_m(A+\epsilon)\right)\right\|/5;$

(ii) $\|H_m(A) - H_m(A+\epsilon)\| \geq \left\|Hess(W_m(A)) - Hess\left(W_m(A+\epsilon)\right)\right\|/2;$

(iii) $\|H_m(A)\| \leq 2K.$

Proof. (i) Indeed since P is harmonic one easily calculates that

$$Tr\left(Hess\left(w^m(a)\right) - Hess\left(w^m(a+\epsilon)\right)\right) = -15\left(w^m(a) - w^m(a+\epsilon)\right)$$

.

(ii) Direct calculations show that

$$Hess\left(W_m^\delta(A)\right) = Hess\left(W_m(A)\right) - \delta\left(\nabla W_m(A) \cdot A^t + A \cdot \nabla^t W_m(A)\right) - \delta W_m(A) I_{24}$$
$$+\delta(\delta+2) W_m(A)(A \cdot A^t),$$
$$Hess\left(r_m^\delta(A)\right) = 2J - \delta r_m(A) I_{24} + N(A)$$

,

where I_n is the identity matrix of size n, and $J, N(A) \in Mat(24 \times 24, \mathbb{R})$ are defined by:

$$J = \begin{pmatrix} I_{12} & 0 \\ 0 & -I_{12} \end{pmatrix}, \quad N(A) = \begin{pmatrix} P(A) & Q(A) \\ R(A) & S(A) \end{pmatrix},$$

with the following matrices $P(A), Q(A), R(A), S(A) \in Mat(12 \times 12, \mathbb{R})$:

$$P(A) = -\delta(4 + 2r_m(A) + \delta r_m(A))(a^t \cdot a),$$

$$S(A) = \delta(4 + 2r_m(A) + \delta r_m(A))(a'^t \cdot a'),$$

$$Q(A) = \delta(2 + \delta)r_m(A))(a'^t \cdot a),$$

$$R(A) = \delta(2 + \delta)r_m(A))(a^t \cdot a').$$

Hence

$$\|H_m(A) - H_m(A+\epsilon)\|$$

$$= \left\| Hess\left(W_m^\delta(A)\right) - Hess\left(W_m^\delta(A+\epsilon)\right) - K\delta r_m(A) I_{24} + KN(A) + K\delta r_m(A+\epsilon) I_{24} - KN(A+\epsilon) \right\|$$

$$\geq \left\| Hess(W_m^\delta(A)) - Hess(W_m^\delta(A+\epsilon)) \right\| - K\delta |r_m(A) - r_m(A+\epsilon)| - K\|N(A) - N(A+\epsilon)\|$$

$$\geq \left\| Hess(W_m(A)) - Hess(W_m(A+\epsilon)) \right\| - \delta \left\| \left(\nabla W_m(A) \cdot A^t + A \cdot \nabla^t W_m(A)\right) - \nabla W_m(A+\epsilon) \cdot (A+\epsilon)^t - (A+\epsilon) \cdot \nabla^t W_m(A+\epsilon) \right\| - \delta |W_m(A) - W_m(A+\epsilon)| - 3\delta \left\| W_m(A)(A \cdot A^t) - W_m(A+\epsilon)\left((A+\epsilon)\cdot(A+\epsilon)^t\right) \right\| - 66\sqrt{2}K\delta |A - (A+\epsilon)|$$

$$\geq \left\| Hess(W_m(A)) - Hess(W_m(A+\epsilon)) \right\| - \delta \|\nabla W_m(A) \cdot A^t - -\nabla W_m(A+\epsilon) \cdot A^t\| - \delta \|\nabla W_m(A+\epsilon) \cdot A^t - \nabla W_m(A+\epsilon) \cdot (A+\epsilon)^t\| - \delta \|A \cdot \nabla^t W_m(A) - (A+\epsilon) \cdot \nabla^t W_m(A)\| - \delta \|(A+\epsilon) \cdot \nabla^t W_m(A) - (A+\epsilon) \cdot \nabla^t W_m(A+\epsilon)\| - 3K\delta \|W_m(A)(A \cdot A^t) - W_m(A+\epsilon)(A \cdot A^t)\| - 3K\delta \left\| W_m(A+\epsilon)\left((A+\epsilon)\cdot A^t\right) - W_m(A+\epsilon)(A \cdot A^t) \right\| - 3K\delta \left\| W_m(A+\epsilon)\left((A+\epsilon)\cdot A^t\right) - W_m(A+\epsilon)\left((A+\epsilon)\cdot(A+\epsilon)^t\right) \right\| - 66\sqrt{2}K\delta |A - (A+\epsilon)|$$

$$\geq \left\| Hess(W_m(A)) - Hess(W_m(A+\epsilon)) \right\| - 2\delta |\nabla W_m(A) - \nabla W_m(A+\epsilon)| - 3\delta |W_m(A) - W_m(A+\epsilon)| - 100K\delta |A - (A+\epsilon)|$$

$$\geq \left(1 - 24\sqrt{3} \cdot 100(K+1)\delta\right) \left\| Hess(W_m(A)) - Hess(W_m(A+\epsilon)) \right\|$$

$$\geq \left\| Hess(W_m(A)) - Hess(W_m(A+\epsilon)) \right\| / 2,$$

since one can easily verify that $\|N(A) - N(A+\epsilon)\| \leq 64\sqrt{2}K\delta |A - (A+\epsilon)|$ and $|\nabla W_m(A) - \nabla W_m(A+\epsilon)| \leq 50|A - (A+\epsilon)|$

(iii).

We have

$$\|H_m(A)\| \leq \left\| Hess\left(W_m^\delta(A)\right) \right\| + K + \delta(7+\delta)$$

$$\leq \left\| Hess\left(W_m(A)\right) \right\| + \delta(2|\nabla W_m(A)| + 12 + 2\delta) + K + \delta(7+\delta)$$

$$\leq 2/\sqrt{3} + 200\delta + K < 2K.$$

Lemma (2.4):

For any pair $A = (a, a'),\ A + \epsilon = (a + \epsilon, a' + \epsilon') \in S_1^{23}$ there exists $E = (e, e') \in S_1^{23}$ with $E \perp A,\ E \perp (A + \epsilon)$ satisfying:

$$W_{K,E,E}^\delta(A + \epsilon) = 0,$$

$$W_{K,E,E}^\delta(A) \geq 2 \cdot 10^{-4} \|H_m(A) - H_m(A + \epsilon)\|.$$

Proof. Define:

$$Q_{A,A+\epsilon}^\delta(E) := W_{K,E,E}^\delta(A) - W_{K,E,E}^\delta(A + \epsilon) = u_{E,E}^m(A) - u_{E,E}^m(A + \epsilon).$$

Note for $E \perp V$ we have:

$$W_{K,E,E}^\delta(V) = \left(W_{E,E}(V) + K(|e|^2 - |e'|^2)\right)|V|^{-\delta} - \delta\left(|X|^2 - |X + \epsilon|^2 + W_m(V)\right)|V|^{-2-\delta}.$$

In particular this remark applies to A and $A + \epsilon$. By Property (ii) we can find: $e_0, e_0' \in S_1^{11}$, $e \perp a$, $e_0' \perp a'$, $e_0 \perp (a + \epsilon)$, $e_0' \perp (a + \epsilon)'$ s.t.

$$\left(w_{e_0,e_0}^m(a) - w_{e_0,e_0}^m(a + \epsilon)\right) \geq \left\| Hess\left(w^m(a)\right) - Hess\left(w^m(a + \epsilon)\right) \right\| / M,$$

$$\left(w_{e_0',e_0'}^m(a') - w_{e_0',e_0'}^m(a + \epsilon)'\right) \geq \left\| Hess\left(w^m(a')\right) - Hess\left(w^m(a + \epsilon)'\right) \right\| / M.$$

Let $\theta \in [0, \pi]$ and let $E(\theta) := \left((\sin\theta)_{e_0}, (\cos\theta)_{e_0'}\right) \in S_1^{23}$; we see that $E(\theta) \perp A$, $E(\theta) \perp A + \epsilon$. One easily verifies that

$$Q^{\delta}_{A,A+\epsilon}(E(\theta)) = u^{m}_{E(\theta),E(\theta)}(A) - u^{m}_{E(\theta),E(\theta)}(A+\epsilon)$$
$$= W_{E(\theta),E(\theta)}(A) - W_{E(\theta),E(\theta)}(A+\epsilon) - \delta\big(W_m(A) - W_m(A+\epsilon)\big)$$
$$-\delta(\sin^2\theta\,|a|^2 - \cos^2\theta\,|a'|^2 - \sin^2\theta\,|a+\epsilon|^2 + \cos^2\theta\,|a'+\epsilon'|^2)$$
$$= \sin^2\theta\left(w^m_{e_0,e_0}(a) - w^m_{e_0,e_0}(a+\epsilon)\right) + \cos^2\theta\left(w^m_{e_0',e_0'}(a') - w^m_{e_0',e_0'}(a+\epsilon)'\right)$$
$$-\delta\big(W_m(A) - W_m(A+\epsilon)\big)$$
$$-\delta\big(\sin^2\theta\,(|a|^2 - |a+\epsilon|^2) + \cos^2\theta\,(|a'|^2 - |a'+\epsilon'|^2)\big).$$

Let now $\theta \in \left[\arcsin\frac{7}{10} = \theta_0 \,, \arccos\frac{7}{10} = \theta_1\right]$.

Then

$$Q^{\delta}_{A,A+\epsilon}(E(\theta))$$
$$\geq (2M^{-1}\sin^2\theta_0 - 8\delta/15)\big(\|Hess(W_m(A)) - Hess(W_m(A+\epsilon))\|\big)$$
$$-2\delta\cos^2\theta_0\,(|a-(a+\epsilon)| + |a'-(a+\epsilon)'|)$$
$$\geq (2M^{-1}0.49 - 8\delta/15)\big(\|Hess(W_m(A)) - Hess(W_m(A+\epsilon))\|\big)$$
$$-1.02\delta(|a-(a+\epsilon)| + |a'-(a+\epsilon)'|)$$
$$\geq (0.98M^{-1} - 50\delta)\big(\|Hess(W_m(A)) - Hess(W_m(a+\epsilon))\|\big)$$
$$\geq 2\cdot 10^{-4}\|H_m(A) - H_m(A+\epsilon)\|.$$

Besides,

$$W^{\delta}_{K,E(\theta),E(\theta)}(A+\epsilon) = W_{E(\theta),E(\theta)}(A+\epsilon) - \delta W_m(A+\epsilon) + K\cos 2\theta$$
$$-\delta(|a+\epsilon|^2 - |a'+\epsilon'|^2).$$

This gives for $\theta = \theta_0$,

$$W^{\delta}_{K,E(\theta_0),E(\theta_0)}(A+\epsilon) = W_{E(\theta_0),E(\theta_0)}(A+\epsilon) - \delta W_m(A+\epsilon) + 0.02K$$
$$-\delta(|a+\epsilon|^2 - |a'+\epsilon'|^2) > -2/\sqrt{3} - 2\delta + 1.2 > 0$$

;

and for $\theta = \theta_1$

$$W^{\delta}_{K,E(\theta_1),E(\theta_1)}(A+\epsilon) = W_{E(\theta_1),E(\theta_1)}(A+\epsilon) - \delta W_m(A+\epsilon) + 0.02K$$
$$-\delta(|a+\epsilon|^2 - |a'+\epsilon'|^2) < 2/\sqrt{3} + 2\delta - 1.2 < 0$$

The lemma follows for $E = E(\theta)$ with $\theta \in \,]\theta_0, \theta_1[$.

Proposition (2.5) (Main Lemma):

For any pair $A = (a, a'),\ A+\epsilon = (a+\epsilon, a'+\epsilon') \in B^{23}_1$ there exist two vectors $E = (e, e'),\ \bar{E} = (\bar{e}, \bar{e}') \in S^{23}_1$ with $E \perp A,\ E \perp (A+\epsilon)$, $\bar{E} \perp A,\ \bar{E} \perp (A+\epsilon)$ satisfying:

$$W^{\delta}_{K,E,E}(A) - W^{\delta}_{K,E,E}(A+\epsilon) \geq \epsilon\|H_m(A) - H_m(A+\epsilon)\|$$

,

$$W^{\delta}_{K,\bar{E},\bar{E}}(A) - W^{\delta}_{K,\bar{E},\bar{E}}(A+\epsilon) \leq -\epsilon\|H_m(A) - H_m(A+\epsilon)\|$$

,

where $\epsilon := 10^{-4}$.

Proof. We can suppose w.r.g. that $|A| \leq |A+\epsilon|$. Since $W^{\delta}_{K,E,E}(A) - W^{\delta}_{K,E,E}(A+\epsilon)$ and $\|H_m(A) - H_m(A+\epsilon)\|$ are both $(-\delta)$-homogeneous one can as well suppose that $|A+\epsilon| = 1$, $1 \geq |A|$.

Define $A' := A/|A| \in S^{23}_1$, $k := |A|$.

We consider two cases:

$$(i)\ \|H_m(A) - H_m(A+\epsilon)\| \leq 2\|H_m(A') - H_m(A+\epsilon)\|$$

;

(ii)

$$\|H_m(A) - H_m(A+\epsilon)\| \geq 2\|H_m(A') - H_m(A+\epsilon)\|$$

.

In the case (i) we apply Lemma 2 and find a vector $E \in S^{23}_1$ such that

$$W^{\delta}_{K,E,E}(A') - W^{\delta}_{K,E,E}(A+\epsilon) = W^{\delta}_{K,E,E}(A) \geq 2\epsilon\|H_m(A') - H_m(A+\epsilon)\|$$
$$\geq \epsilon\|H_m(A) - H_m(A+\epsilon)\|$$

The second inequality is obtained analogously.

Let now

$$\|H_m(A) - H_m(A+\epsilon)\| \geq 2\|H_m(A') - H_m(A+\epsilon)\|$$

.

Since

$$\|H_m(A) - H_m(A+\epsilon)\| = \left\|k^{-\delta}H_m(A') - H_m(A+\epsilon)\right\|$$
$$\geq 2\|H_m(A') - H_m(A+\epsilon)\|$$

we get:

$$(k^{-\delta} - 1)\|H_m(A')\| \geq \|H_m(A') - H_m(A+\epsilon)\|$$

.

Thus,

$$\|H_m(A) - H_m(A+\epsilon)\| = \left\|k^{-\delta}H_m(A') - H_m(A+\epsilon)\right\|$$
$$= \left\|(k^{-\delta} - 1)H_m(A') + H_m(A') - H_m(A+\epsilon)\right\|$$

$$\le \left(k^{-\delta}-1\right)\|H_m(A')\| + \|H_m(A') - H_m(A+\epsilon)\|$$
$$\le 2\left(k^{-\delta}-1\right)\|H_m(A')\|$$

Take now a vector $E' = (e', 0)$ such that

$$W^{\delta}_{K,E',E'}(A') - W^{\delta}_{K,E',E'}(A+\epsilon)$$
$$= W^{\delta}_{E',E'}(A') - W^{\delta}_{E',E'}(A+\epsilon) + K\cdot r^{\delta}_{E',E'}(A') - K\cdot r^{\delta}_{E',E'}(A+\epsilon) \ge 0$$

We then get,

$$W^{\delta}_{K,E',E'}(A') - W^{\delta}_{K,E',E'}(A+\epsilon)$$
$$= \left(k^{-\delta}-1\right)W^{\delta}_{K,E',E'}(A') + W^{\delta}_{K,E',E'}(A') - W^{\delta}_{K,E',E'}(A+\epsilon)$$
$$\ge \left(k^{-\delta}-1\right)\left(w^{m}_{e',e'}(a') + K - \delta(|a'|^2 - |a''|^2 + W_m(A'))\right)$$
$$\ge \left(k^{-\delta}-1\right)(K - 8 - 3\delta) \ge \left(k^{-\delta}-1\right)K/2$$
$$\ge \left(k^{-\delta}-1\right)\|H_m(A')\|/4 \ge \|H_m(A) - H_m(A+\epsilon)\|/8$$

which finishes the proof of the first inequality; the proof of the second one is completely parallel.

III. VISCOSITY SOLUTIONS OFUNIFORMLY ELLIPTIC EQUATIONS ON $\mathbb{R}^{24}$

Theorem (3.1):

We provethat, for $\delta = 10^{-6}$ there exists a continuous homogeneous order $2-\delta$ function u^m in the unit ball $B \subset \mathbb{R}^{24}$ which is a viscosity solution to a uniformly elliptic equation (1.1).

Notice, that there are no defined in the whole space $\mathbb{R}^n$ homogeneous order α solutions to fully nonlinear elliptic equation (1.1) for $0 < \alpha < 2$, [6]. The proof of

Theorem is strongly based on results and methods of [5].

Proof:

Let Q be the space of the quadratic forms on $\mathbb{R}^n$ equipped by its natural inner product $a\cdot(a+\epsilon) = trace(a(a+\epsilon))$ for $a, a+\epsilon \in Q$.

Let us choose in the space Q an orthogonal coordinate system $z_1, z_2, \dots, z_k, s, k = \frac{n(n+1)}{2} - 1$ such that s is the trace. Let $\pi: Q \to Z$ be the orthogonal projection of Q onto the z-space. For $\epsilon > 0$, we denote by $K_{1+\epsilon}$ the cone:

$$K_{1+\epsilon} = \{a \in Q: \exists\, C > 0\ s.t.\ \text{the eigenvalues of } a \in [C/(1+\epsilon), C(1+\epsilon)]\}$$

.

Since on Q the maximal eigenvalue of a quadratic form is a convex function and the minimal eigenvalue is a concave function it follows that $K_{1+\epsilon}$ is a convex cone.

Let $K^*_{1+\epsilon}$ denote the adjoint cone of $K_{1+\epsilon}$, that is,

$$K^*_{1+\epsilon} = \{a+\epsilon \in Q:\ (a+\epsilon)\cdot c = 0\ \text{ for all } c \in K_{1+\epsilon}\}$$

.

As an adjoint to a convex cone the cone $K^*_{1+\epsilon}$ is a convex itself [8].

The Set $L_{1+\epsilon} = Q\backslash(K^*_{1+\epsilon} \cup -K^*_{1+\epsilon})$Notice that $a \in L_{1+\epsilon}$ is equivalent to $a\cdot(a+\epsilon) = 0$ for some $a+\epsilon \in K_{1+\epsilon}$, i.e., $L_{1+\epsilon}$ is a union of all hyper-planes in Q with normals in $K_{1+\epsilon}$.

Let $G \subset Q$ be a set. We say that Gsatisfies the $(a+\epsilon)$-cone condition if for any two points $a, a+\epsilon \in G$, the matrix $a-(a+\epsilon) \in L_{1+\epsilon}$.

Lemma (3.2):

Let $\Sigma^m \subset Q$ be a smooth k-dimensional manifold. Assume that Σ satisfies the $(1+\epsilon)$-cone condition. Then there exists a smooth function F^m on Q such that $\mathrm{F}^m(\Sigma^m) = 0$, and which satisfies the inequality (2) with the ellipticity constant $1+\epsilon < 4(1+\epsilon)^2\sqrt{n}$.

Denote $D = S^{11} \times \left(0, 1/\sqrt{2}\right),\ G = D^2$. Define a map $f_m: D \to B^{12}$ and $g_m: G \to B^{24}$ such that if $a \in S^{11}, \theta \in (0,1)$, $x \in (a,\theta)$ then $f_m(x) = \theta a$, and if $z_1, z_2 \in D$, $z = (z_1, z_2) \in G$ then $g_m(z) = (f_m(z_1), f_m(z_2))$. The Hessian map H_m

for the function u^m is defined on the set $B^{24}\setminus(\{X=0\}\cup\{X+\epsilon=0\})$:
$H_m: B\to Q,\quad H_m(A):=Hess(u^m(A))$
for $A\in B^{24}\setminus(\{X=0\}\cup\{X+\epsilon=0\})$, Q begin the space of the quadratic forms on $\mathbb{R}^{24}$.
Since
$g_m(G)\subset B^{24}\setminus(\{X=0\}\cup\{X+\epsilon=0\})$ we can lift H_m on G: for $z\in G$ define
$h_m(z)=H_m(g_m(z))$.
Since w^m is a homogeneous order 2 function on $\mathbb{R}^{12}$ we conclude that the map
$h_m: G\to Q$
has a smooth extension to
$h_m: \bar{G}-E\to Q$,
where $E:=(\{0\}\times S_1^{11})\cup(S_1^{11}\times\{0\})$. Denote:
$\Sigma^m=h_m(\bar{G}-E)$.
Then Σ^m is a closed manifold with boundary in Q. By Main Lemma Σ^m satisfies the $(1+\epsilon)$-cone condition with $1+\epsilon=23\cdot10^4$. Hence by Lemma 3 there exists a smooth function F^m on Q which satisfies the inequalities$(*)$ with the ellipticity constant $1+\epsilon<4\cdot\sqrt{24}\cdot23^2\cdot10^8<1.1\cdot10^{12}$ and such that $F^m_{|\Sigma^m}=0$. Thus for $z\in B^{24}\cap(\{X=0\}\cup\{X+\epsilon=0\})$ we have:
$F^m(D^2u^m(z))=0.$
To complete the proof that u^m is a viscosity solution of (1.1) it is sufficient to show that for any point $z_0\in B^{24}\cap(\{X=0\}\cup\{X+\epsilon=0\})$ and for second order polynomials $p_1(p_2)$ on $\mathbb{R}^{24}$ such that $p_1(z_0)=p_2(z_0)=u^m(z_0)$ and such that $p_1\le u^m(p_2\ge u^m)$ in a neighborhood of z_0 it will follow that $F^m(D^2p_1)\le 0(F^m(D^2p_2)\ge 0)$.
Let $z_0=(0,x+\epsilon)\in\mathbb{R}^{24}, e\in S^{23}$. Since w^m is a homogeneous order 2 function in $\mathbb{R}^{12}\setminus\{0\}$ it follows that $u^m(z_0+\epsilon e)$ is a smooth function for $\epsilon\ge0$.

We define a homogeneous order 2 function ψ^m on $\mathbb{R}^{24}$ such that for any $e\in S^{23}$ the quadratic part of $u^m(z_0+\epsilon c)$ as a function of ϵ coincide with $\psi^m(\epsilon e)$. Since the range of $Hess(\psi^m)$ coincide with the limit set of $Hess(u^m(z))$ as $z\to z_0$, $z\in B$ it follows that ψ^m is a solution of the equation
$F^m(D^2\psi^m)=0$.
Let $p_m(x),\ x\in\mathbb{R}^{24}$ be a quadratic form such that $p_m\le w^m$ on $\mathbb{R}^{24}$. We choose any quadratic form $p'_m(x)$ such that $p_m\le p'_m\le\psi^m$ and there is a point $x'\ne0$ at which $p'_m(x')=\psi^m(x')$. Then it follows that $F^m(p_m)\le F^m(p'_m)\le 0$. Consequently for any quadratic form $p_m(x)$ from the inequality $p_m\le\psi^m(p_m\ge\psi^m)$ it follows that $F^m(p_m)\le 0(F^m(p_m)\ge 0)$. This implies that ψ^m is a viscosity solution of (1.1) in $\mathbb{R}^{24}$ (see [1]).

Corollary (3.3):
For any pair $A=(a,a'),A+\varepsilon=(a+\varepsilon,a'+\varepsilon')\in S_1^{23}$ there exists $E=(e,e')\in S_1^{23}$ with $E\perp A, E\perp A+\varepsilon$ satisfying:
$W^\delta_{K,E,E}(A+\varepsilon)=0$,
$W^\delta_{K,E,E}(A)\ge 2\cdot10^{-4}\|H(A)-H(A+\varepsilon)\|$.

Proof: Define:
$Q^\delta_{A,B}(E):=W^\delta_{K,E,E}(A)-W^\delta_{K,E,E}(A+\varepsilon)=u_{E,E}(A)-u_{E,E}(A+\varepsilon)$
.
Note for $E\perp V$ we have:
$W^\delta_{K,E,E}(V)=(W_{E,E}(V)+K(|e|^2-|e'|^2))|V|^{-\delta}-\delta(|X|^2-|Y|^2+W(V))|V|^{-2-\delta}$
.
In particular this remark applies to A and $A+\varepsilon$. By Property (ii) in Lemma (2.3) we can find:
$e_0,e'_0\in S_1^{11}, e\perp a, e'_0\perp a', e_0\perp a+\varepsilon, e'_0\perp a'+\varepsilon'$
s.t.

$(w_{e_0,e_0}(a) - w_{e_0,e_0}(a+\varepsilon)) \geq \|\mathrm{Hess}(w(a)) - \mathrm{Hess}(w(a+\varepsilon))\|/M$,

$(w_{e_0',e_0'}(a') - w_{e_0',e_0'}(a'+\varepsilon')) \geq \|\mathrm{Hess}(w(a')) - \mathrm{Hess}(w(a'+\varepsilon'))\|/M$.

Let $\theta \in [0,\pi]$ and let $E(\theta) := ((\sin\theta)e_0, (\cos\theta)e_0') \in S_1^{23}$; we see that $E(\theta) \perp A, E(\theta) \perp A+\varepsilon$. One easily verifies that

$$Q_{A,B}^{\delta}(E(\theta)) = u_{E(\theta),E(\theta)}(A) - u_{E(\theta),E(\theta)}(A+\varepsilon)$$
$$= W_{E(\theta),E(\theta)}(A) - W_{E(\theta),E(\theta)}(A+\varepsilon) - \delta(W(A) - W(A+\varepsilon))$$
$$-\delta(\sin^2\theta\,|a|^2 - \cos^2\theta\,|a'|^2 - \sin^2\theta\,|a+\varepsilon|^2 + \cos^2\theta\,|a'+\varepsilon'|^2)$$
$$= \sin^2\theta\,(w_{e_0,e_0}(a) - w_{e_0,e_0}(a+\varepsilon)) + \cos^2\theta\,(w_{e_0',e_0'}(a') - w_{e_0',e_0'}(a'+\varepsilon')) - \delta(W(A) - W(A+\varepsilon))$$
$$-\delta(\sin^2\theta\,(|a|^2 - |a+\varepsilon|^2) + \cos^2\theta\,(|a'|^2 - |a'+\varepsilon'|^2))$$.

Let now $\theta \in [\arcsin\frac{7}{10} = \theta_0, \arccos\frac{7}{10} = \theta_1]$.
Then

$$Q_{A,B}^{\delta}(E(\theta)) \geq (2M^{-1}\sin^2\theta_0 - 8\delta/15)(\|\mathrm{Hess}(W(A)) - \mathrm{Hess}(W(A+\varepsilon))\|)$$
$$-2\delta\cos^2\theta_0\,(|a-(a+\varepsilon)| + |a'-(a'+\varepsilon')|)$$
$$\geq (2M^{-1}0.49 - 8\delta/15)(\|\mathrm{Hess}(W(A)) - \mathrm{Hess}(W(A+\varepsilon))\|)$$
$$-1.02\delta(|a-(a+\varepsilon)| + |a'-(a'+\varepsilon')|)$$
$$\geq (0.98M^{-1} - 50\delta)(\|\mathrm{Hess}(W(A)) - \mathrm{Hess}(W(A+\varepsilon))\|)$$
$$\geq 2\cdot 10^{-4}\|H(A) - H(A+\varepsilon)\|.$$

Besides,

$$W_{K,E(\theta),E(\theta)}^{\delta}(A+\varepsilon) = W_{E(\theta),E(\theta)}(A+\varepsilon) - \delta W(A+\varepsilon) + K\cos 2\theta - \delta(|a+\varepsilon|^2 - |a'+\varepsilon'|^2).$$

This gives for $\theta = \theta_0$,

$$W_{K,E(\theta_0),E(\theta_0)}^{\delta}(A+\varepsilon) = W_{E(\theta_0),E(\theta_0)}(A+\varepsilon) - \delta W(A+\varepsilon) + 0.02K - \delta(|a+\varepsilon|^2 - |a'+\varepsilon'|^2) > -2/\sqrt{3} - 2\delta + 1.2 > 0$$;

and for $\theta = \theta_1$

$$W_{K,E(\theta_1),E(\theta_1)}^{\delta}(A+\varepsilon) = W_{E(\theta_1),E(\theta_1)}(A+\varepsilon) - \delta W(A+\varepsilon) - 0.02K - \delta(|a+\varepsilon|^2 - |a'+\varepsilon'|^2) < 2/\sqrt{3} + 2\delta - 1.2 < 0$$.

The corollary follows for $E = E(\theta)$ with $\theta \in]\theta_0, \theta_1[$.

Corollary (3.4): For any pair $A = (a,a'), A+\varepsilon = (a+\varepsilon, a'+\varepsilon') \in B_1^{23}$ there exist two vectors $E = (e,e'), \bar{E} = (\bar{e},\bar{e}') \in S_1^{23}$ with $E \perp A, E \perp A+\varepsilon, \bar{E} \perp A, \bar{E} \perp A+\varepsilon$ satisfying:

$$W_{K,E,E}^{\delta}(A) - W_{K,E,E}^{\delta}(A+\varepsilon) \geq \varepsilon\|H(A) - H(A+\varepsilon)\|$$,

$$W_{K,\bar{E},\bar{E}}^{\delta}(A) - W_{K,\bar{E},\bar{E}}^{\delta}(A+\varepsilon) \leq -\varepsilon\|H(A) - H(A+\varepsilon)\|$$,

where $\varepsilon := 10^{-4}$.

Proof: We can suppose w.r.t. that $|A| \leq |A+\varepsilon|$. Since $W_{K,E,E}^{\delta}(A) - W_{K,E,E}^{\delta}(A+\varepsilon)$ and $\|H(A) - H(A+\varepsilon)\|$ are both $(-\delta)$-homogeneous one can as well suppose that $|A+\varepsilon| = 1, 1 \geq |A|$.

Define $A' := A/|A| \in S_1^{23}, k := |A|$.

We consider two cases:

(i) $\|H(A) - H(A+\varepsilon)\| \leq 2\|H(A') - H(A+\varepsilon)\|$;

(ii) $\|H(A) - H(A+\varepsilon)\| \geq 2\|H(A') - H(A+\varepsilon)\|$.

In the case (i) we apply Lemma (5.1.4) and find a vector $E \in S_1^{23}$ such that

$$W_{K,E,E}^{\delta}(A') - W_{K,E,E}^{\delta}(A+\varepsilon) = W_{K,E,E}^{\delta}(A)$$
$$\geq 2\varepsilon\|H(A') - H(A+\varepsilon)\| \geq \varepsilon\|H(A) - H(A+\varepsilon)\|$$

The second inequality is obtained analogously. Let now

$$\|H(A) - H(A+\varepsilon)\| \geq 2\|H(A') - H(A+\varepsilon)\|$$.

Since

$\|H(A)-H(B)\| = \|k^{-\delta}H(A')-H(B)\| \geq 2\|H(A')-H(B)\|$

, we get:

$(k^{-\delta}-1)\|H(A')\| \geq \|H(A')-H(B)\|$.

Thus,

$\|H(A)-H(A+\varepsilon)\| = \|k^{-\delta}H(A')-H(A+\varepsilon)\| = \|(k^{-\delta}-1)H(A')+H(A')-H(A+\varepsilon)\| \leq (k^{-\delta}-1)\|H(A')\| + \|H(A')-H(A+\varepsilon)\| \leq 2(k^{-\delta}-1)\|H(A')\|$

.

Take now a vector $E'=(e',0)$ such that

$W^{\delta}_{K,E',E'}(A') - W^{\delta}_{K,E',E'}(A+\varepsilon) = W^{\delta}_{E',E'}(A') - W^{\delta}_{E',E'}(A+\varepsilon) + K\cdot r^{\delta}_{E',E'}(A') - K\cdot r^{\delta}_{E',E'}(A+\varepsilon) \geq 0$.

We then get,

$W^{\delta}_{K,E',E'}(A') - W^{\delta}_{K,E',E'}(A+\varepsilon)$
$= (k^{-\delta}-1)W^{\delta}_{K,E',E'}(A') + W^{\delta}_{K,E',E'}(A') - W^{\delta}_{K,E',E'}(A+\varepsilon)$
$\geq (k^{-\delta}-1)(w_{e',e'}(a') + K - \delta(|a'|^2 - |a''|^2 + W(A')))$
$\geq (k^{-\delta}-1)(K-8-3\delta) \geq (k^{-\delta}-1)K/2 \geq (k^{-\delta}-1)\|H(A')\|/4$
$\geq \|H(A)-H(A+\varepsilon)\|/8$

which finishes the proof of the first inequality; the proof of the second one is completely parallel.

Corollary (3.5):

For $\delta = 10^{-6}$ there exists a continuous homogeneous order $2-\delta$ function u_j in the unit ball $B \subset \mathbb{R}^{24}$ which is a viscosity solution to a uniformly elliptic equation (1), i.e., $F(D^2u_j)=0$

Proof: Let Q be the space of the quadratic forms on $\mathbb{R}^n$ equipped by its natural inner product $a\cdot(a+\varepsilon) = \mathrm{trace}(a(a+\varepsilon))$ for $a \in Q$ and $\varepsilon > 0$. Let us choose in the space Q an orthogonal coordinate system $z_1,\ldots,z_k,s,k=\frac{n(n+1)}{2}-1$ such that s is the trace. Let $\pi: Q \to Z$ be the orthogonal projection of Q onto the z-space. For $\varepsilon > 0$ we denote by $K_{1+\varepsilon}$ the cone:

$K_{1+\varepsilon} = \{a \in Q\text{:there exists}C > 0\text{s.t.the eigenvalues of}a \in [C/(1+\varepsilon), C(1+\varepsilon)]\}$

.

Since on Q the maximal eigenvalue of a quadratic form is a convex function and the minimal eigenvalue is a concave function it follows that K_Λ is a convex cone. Let $K^*_{1+\varepsilon}$ denote the adjoint cone of $K_{1+\varepsilon}$, that is,

$K^*_{1+\varepsilon} = \{a+\varepsilon \in Q\text{: }(a+\varepsilon)\cdot c \geq 0\text{for all}c \in K_{1+\varepsilon}\}$

. As an adjoint to a convex cone the cone $K^*_{1+\varepsilon}$ is a convex itself. Set $L_{1+\varepsilon} = Q\backslash(K^*_{1+\varepsilon} \cup -K^*_{1+\varepsilon})$. Notice that $a \in L_{1+\varepsilon}$ is equivalent to $a\cdot(a+\varepsilon)=0$ for some $a+\varepsilon \in K_{1+\varepsilon}$, i.e., $L_{1+\varepsilon}$ is a union of all hyper-planes in Q with normals in $K_{1+\varepsilon}$. Let $G \subset Q$ be a set. We say that G satisfies the $(1+\varepsilon)$-cone condition if for any two points $a, a+\varepsilon \in G$, the matrix $a-(a+\varepsilon) \in L_{1+\varepsilon}$.

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