Lie Nilpotency Index of Group Algebra

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Abstract

Let G be a group and let K be a field of characteristic $p > 0$. In this paper, we have focused on Lie nilpotency index of group algebra.

1 Introduction

Let KG be the group algebra of a group G over a field K of characteristic $p > 0$. KG can be regarded as a Lie algebra under the Lie multiplication $[a, b] = ab - ba$ for all $a, b \in KG$. Let $KG^{[1]} = KG$ and for $n > 1$, the nth lower Lie power $KG^{[n]}$ of KG is the associative ideal generated by all the Lie commutators $[x_1, x_2, ..., x_n]$, where $x_1, x_2, ..., x_n \in KG$. We define the nth strong Lie power $KG^{(n)}$ of KG as the associative ideal generated by all the Lie commutators $[x, y]$, where $KG^{(1)} = KG$, $x \in KG^{(n-1)}$ and $y \in KG$. Group algebra KG is called Lie nilpotent (strongly Lie nilpotent) if $KG^{[n]} = 0$ ($KG^{(n)} = 0$) for some $n \in N$ and the least non negative integer n such that $KG^{[n]} = 0$ ($KG^{(n)} = 0$) is called the Lie nilpotency index (strong Lie nilpotency index) of KG and is denoted by $t_L(KG)$ ($t^L(KG)$).

Necessary and sufficent conditions for a group algebra to be Lie nilpotent is given by Bovdi, Khripta, Passi, Passman and Sehgal [4, 10]. They proved that the following statements are equivalent:

1. KG is Lie nilpotent;
2. KG is strongly Lie nilpotent;
3. $CharK = p > 0$, G is nilpotent and its commutator subgroup G' is a finite p-group.

Bhandari and Passi [1], established that for a Lie nilpotent group algebra KG, $t_L(KG) = t^L(KG)$, if $p \geq 5$. But the question whether $t_L(KG) = t^L(KG)$, is in general still open. After that a lot of work has been done on Lie nilpotent group algebras. In this paper, we have focused on Lie nilpotency index of group algebra. We have used the notation $S(n, m)$ for Small group number m of order n from the Small Groups Library-GAP.

2 Preliminaries

Let K be a field of characteristic $p > 0$ and let G be a group. The subgroup $D_{(m),K}(G) = G \cap (1 + KG^{(m)})$, $m \geq 1$ is called the m-th Lie dimension subgroup of KG. By [9, p.48,Theorem 2.8], we have

$$D_{(m+1),K}(G) = \begin{cases} G & \text{if } m = 0 \\ G & \text{if } m = 1 \\ (D_{(m),K}(G), G)(D_{(\lceil \frac{m}{p} \rceil+1),K}(G))^p & \text{if } m \geq 2 \end{cases}$$

Here$\lceil \frac{m}{p} \rceil$ is the upper integral part of $\frac{m}{p}$. These subgroups play an important role in the computation of strong Lie nilpotency indices. If KG is Lie nilpotent such that $|G'| = p^n$, then according to Jennings' theory [14],

$$t^L(KG) = 2 + (p-1)\left(\sum_{m\geq 1} md_{(m+1)}\right) \tag{1}$$

where $p^{d_{(m)}} = [D_{(m),K}(G) : D_{(m+1),K}(G)]$. It is easy to see that $\sum_{m\geq 2} d_{(m)} = n$.

Lemma 2.1. *[15, 16] Let K be a field with characteristic $K = p > 0$ and let G be a nilpotent group such that $|G'| = p^n$ and $exp(G') = p^l$.*

1. *If $d_{(m+1)} = 0$ and m is a power of p, then $D_{(m+1),K}(G) = 1$.*
2. *If $d_{(m+1)} = 0$ and p^{l-1} divides m, then $D_{(m+1),K}(G) = 1$.*
3. *If $p \geq 5$ and $t_L(KG) < p^n + 1$, then $t_L(KG) \leq p^{n-1} + 2p - 1$.*
4. *If $d_{(l+1)} = 0$ for some $l < pm$, then $d_{(pm+1)} \leq d_{(m+1)}$.*
5. *If $d_{(m+1)} = 0$, then $d_{(l+1)} = 0$ for all $l \geq m$ such that $v_{p'}(l) \geq v_{p'}(m)$ where $v_{p'}(x)$ is the maximal divisor of x which is relatively prime to p.*

3 Main Results

Lie nilpotent group algebras play significant role. A lot of work has been done in this area. Bhandari and Passi [1], established that for a Lie nilpotent group algebra KG, $t_L(KG) = t^L(KG)$, if $p \geq 5$. But the question whether $t_L(KG) = t^L(KG)$, is in general still open. In 1990, Sharma and Srivastava proved the following result

Theorem 3.1. *[18] Let KG be a Lie nilpotent group algebra of a group G over a field K of characteristic $p > 0$. Then for $p \geq t_L(KG)$, G is an abelian group.*

In 1992, Sharma and Bisht [17] provided an upper bound for the Lie nilpotency index. They proved that if KG is Lie nilpotent then $t_L(KG) \leq t^L(KG) \leq |G'| + 1$.

A Lie nilpotent group algebra KG has

maximal strong Lie nilpotency index, if $t^L(KG) \leq |G'| + 1$

maximal Lie nilpotency index, $t_L(KG) \leq |G'| + 1$

almost maximal strong Lie nilpotency index, $t^L(KG) \leq |G'| - p + 2$

almost maximal Lie nilpotency index, $t_L(KG) \leq |G'| - p + 2$

Shalev started the study of Lie nilpotent group algebras having maximal strong Lie nilpotency index $|G'| + 1$. This study was completed by Bovdi and Spinelli by proving the following result.

Theorem 3.2. *[6] Let KG be a Lie nilpotent group algebra of a group G over a field K of characteristic* $p > 0$. *Then* $t_L(KG) = |G'| + 1$, *if and only if one of the following conditions holds:*

1. G' *is cyclic;*
2. $p = 2$, $G' \simeq C_2 \times C_2$ *and* $\gamma_3(G) \neq 1$.

Moreover, $t_L(KG) = |G'| + 1$ *if and only if* $t^L(KG) = |G'| + 1$.

In [3], authors characterized the group algebra having almost maximal strong Lie nilpotency index i.e. $t^L(KG) \leq |G'| - p + 2$. They proved the following result

Theorem 3.3. *[3] Let KG be a Lie nilpotent group algebra over a field K of positive characteristic p. Then KG has almost maximal strong Lie nilpotency index if and only if one of the following conditions holds:*

1. $p = 2$, $cl(G) = 2$ *and* $\gamma_2(G)$ *is noncyclic of order* 4*;*
2. $p = 2$, $cl(G) = 4$, $\gamma_2(G) = C_4 \times C_2$ *and* $\gamma_3(G) = C_2 \times C_2$*;*
3. $p = 2$, $cl(G) = 4$, $\gamma_2(G)$ *is elementary abelian of order* 8*;*
4. $p = 3$, $cl(G) = 3$ *and* $\gamma_2(G)$ *is elementary abelian of order* 9.

After this a classification of group algebra having almost maximal Lie nilpotency index is given in [2].

Theorem 3.4. *[2] Let KG be a Lie nilpotent group algebra over a field K of positive characteristic p. Then KG has almost maximal Lie nilpotency index if and only if one of the following conditions holds:*

1. $p = 2$, $cl(G) = 2$ *and* $\gamma_2(G)$ *is noncyclic of order*4*;*
2. $p = 2$, $cl(G) = 4$, $\gamma_2(G) = C_4 \times C_2$ *and* $\gamma_3(G) = C_2 \times C_2$*;*
3. $p = 2$, $cl(G) = 4$, $\gamma_2(G)$ *is elementary abelian of order* 8*;*
4. $p = 3$, $cl(G) = 3$ *and* $\gamma_2(G)$ *is elementary abelian of order* 9.

In the same paper it is also shown that the group algebra KG has almost maximal Lie nilpotency index if and only if it has almost maximal strong Lie nilpotency index.

Lie nilpotent group algebras with $t^L(KG) = |G'| - 2p + 3$, $|G'| - 3p + 4$, $|G'| - 4p + 5$ have been classified in [7]. In the same paper it is also proved that for $k = t_L(KG) = |G'| - 2p + 3, |G'| - 3p + 4, |G'| - 4p + 5$ $t_L(KG) = k$ if and only if $t^L(KG) = k$.

Theorem 3.5. *Let KG be a Lie nilpotent group algebra over a field K of positive characteristic p. Then* $t^L(KG) = |G'| - 4p + 5$ *if and only if one of the following conditions holds:*

1. $p = 2, cl(G) = 2$ *and* $G' = C_2 \times C_2 \times C_2$;
2. $p = 5, cl(G) = 2$ *and* $G' = C_5 \times C_5$.

Moreover, in this case, $t^L(KG) = t_L(KG)$

Theorem 3.6. *LetKG be a Lie nilpotent group algebra over a field K of positive characteristic p. Then* $t^L(KG) = |G'| - 3p + 4$ *if and only if one of the following conditions holds:*

1. $p = 2, cl(G) = 3, G' \simeq C_2 \times C_2 \times C_2$ *and* $\gamma_3(G)$ *is cycli;.*
2. $p = 2, G' \simeq C_4 \times C_2$ *and* $\gamma_2(G)^2 \subseteq \gamma_3(G)$;
3. $p = 5, cl(G) = 3$ *and* $G' \simeq C_5 \times C_5$.

Moreover, in this case, $t^L(KG) = t_L(KG)$

Theorem 3.7. *Let KG be a Lie nilpotent group algebra over a field K of positive characteristic p. Then* $t^L(KG) = |G'| - 2p + 3$ *if and only if one of the following conditions holds:*

1. $p = 2, cl(G) = 3, G' \simeq C_2 \times C_2 \times C_2$ *and* $\gamma_3(G) = C_2 \times C_2$;
2. $p = 2, G' = <a> \times <b> \simeq C_4 \times C_2$ *and* $\gamma_3(G)$ *is one of the groups* $<b>, <a^2> \times <b>, <a^2b>$;
3. $p = 3$, $cl(G) = 3$ *and* $G' = C_3 \times C_3$.

Moreover, in this case, $t^L(KG) = t_L(KG)$.

Theorem 3.8. *Let KG be a Lie nilpotent group algebra over a field K of positive characteristic p. Then:*

1. $t_L(KG) = |G'| - 4p + 5$ *if and only if G and K satisfy one of the conditions of Theorem 3.5.*
2. $t_L(KG) = |G'| - 3p + 4$ *if and only if G and K satisfy one of the conditions of Theorem 3.6.*

3. $t_L(KG) = |G'| - 2p+3$ *if and only if G and K satisfy one of the conditions of Theorem* 3.7.

After this Siwach, Sahai and Sharma, [11], [13], classified Lie nilpotent group algebras with $t^L(KG) = |G'| - 5p + 6$, $|G'| - 6p + 7$, $|G'| - 7p + 8$, $|G'| - 8p + 9$, $|G'| - 9p+10$, $|G'| - 10p+11$, $|G'| - 11p+12$, $|G'| - 12p+13$, or $|G'| - 13p+14$. Working in the same direction several authors such as Sahai, Sharan have done work in this field.

It is clear from the above discussion that Lie nilpotent group algebras have been studied deeply. A lot of work has been done in this field. But there are still more to do.

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