Characterizations of Secondary Nilpotent Matrices

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Abstract- *The concept of secondary nilpotent matrices is introduced. Characterization of secondary nilpotent matrices are obtained and derived some theorems.*

Keywords- Transpose of a matrix, conjugate transpose of a matrix, Secondary transpose of a matrix, nilpotent matrix, nilpotent matrix.

I. INTRODUCTION

Anna Lee [1] has initiated the study of secondary symmetric matrices. Also she has shown that for a complex matrix A, the usual transpose A^T and secondary transpose A^S are related as $A^S=VA^TV$ [2] where 'V' is the permutation matrix with units in its secondary diagonal.Pei Yuan Wu [3] showed that any complex singular square matrix T is a product of two nilpotent matrices A and B with rank A = rank B = rank T except when T is a 2 x 2 nilpotent matrix of rank one. Ram Niwas Gupta [4] showed that over a division ring D, which is not commutative, there exists a 2 x2 matrix A which is nilpotent and whose transpose is invertible. Matrices over general rings which are sums of nilpotent matrices and showed that over commutative rings all matrices with nilpotent trace are sums of three nilpotent matrices was derived by Simion Breaz [5] .

In this paper the s-nilpotent matrices is defined and its characterizations are discussed. Some theorems relating to s-nilpotent matrices are derived.

II. PRELIMINARIES AND NOTATIONS

Let C_{nxn} be the space of nxn complex matrices of order n. For $A \in C_{nxn}$. Let $A^T, \overline{A}, A^*, A^S, A^\theta$ denote transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose of a matrix A respectively. Let 'V' be the associated permutation matrix whose elements on the secondary diagonal are 1, other elements are zero. Also 'V' satisfies the following properties .
$V^T = \overline{V} = V^* = V^{-1} = V, V^2 = I$ and $VV^* = V^*V = I$

III. SECONDARY NILPOTENT MATRICES

3.1 PROPERTIES OF S-NILPOTENT MATRICES

DEFNITION 3.1

A is a secondary nilpotent matrix if $A^p V = 0$ where p is a postive integer and 0 is the zero matrix

THEOREM 3.1.1

If $A \in C_{nxn}$ is a s-nilpotent matrix then A^T is also s-nilpotent matrix.

PROOF:

We know that A is a s-nilpotent matrix then by definition,

$$A^p V = 0$$

Taking transpose on both sides,

$$(A^pV)^T = 0^T$$
$$\Rightarrow V^T (A^P)^T = 0$$
$$\Rightarrow V (A^P)^T = 0 \qquad \{\because V^T = V\}$$

Pre multiplying and post multiplying by V,

$$V^2 (A^P)^T V = V0V$$
$$\Rightarrow (A^P)^T V = 0 \qquad \{\because V^2=I\}$$
$$\Rightarrow (A^T)^P V = 0$$

Therefore A^T is a s- nilpotent matrix.

THEOREM 3.1.2

If $A \in C_{nxn}$ is a s-nilpotent matrix then A^* is a s-nilpotent matrix.

PROOF:

We know that A is a s-nilpotent matrix then by definition,

$$A^p V = 0$$

Taking conjugate transpose on both sides.

$$(A^pV)^* = 0^*$$
$$\Rightarrow \quad V^* (A^P)^* = 0$$
$$\Rightarrow \quad V (A^P)^* = 0 \qquad \{\because V^* = V\}$$

Pre multiplying and post multiplying by V,

$$V^2 (A^P)^* V = V0V$$
$$\Rightarrow (A^P)^* V = 0$$
$$\Rightarrow (A^*)^P V = 0$$

Therefore A* is a s- nilpotent matrix.

THEOREM 3.1.3

If $A \in C_{nxn}$ is a s-nilpotent matrix then A^{-1} is a s-nilpotent matrix.

PROOF:

We know that A is a s-nilpotent matrix then by definition,

$$A^p V = 0$$

Taking inverse on both sides ,

$$(A^pV)^{-1} = 0^{-1}$$
$$\Rightarrow V^{-1} (A^P)^{-1} = 0$$
$$\Rightarrow V (A^P)^{-1} = 0 \qquad \{\because V^{-1} = V\}$$
$$\Rightarrow V(A^{-1})^P = 0$$

Pre multiplying and post multiplying by V,

$$V^2 (A^{-1})^p V = V0V$$
$$\Rightarrow (A^{-1})^P V = 0$$
$$\Rightarrow (A^{-1})^P V = 0$$

Therefore $A^{-1}$is a s- nilpotent matrix.

THEOREM 3.1.4

If $A \in C_{nxn}$ is a s-nilpotent matrix then A^s is a s-nilpotent matrix.

PROOF:

We know that A is a s-nilpotent matrix then by definition,

$$A^p V = 0$$

Taking secondary transpose on both sides.

$$(A^pV)^s = 0^s$$
$$\Rightarrow V^s (A^P)^s = 0$$
$$\Rightarrow V (A^P)^s = 0 \qquad \{\because V^s = V\}$$

Pre multiplying and post multiplying by V,

$$V^2 (A^P)^s V = V0V$$
$$\Rightarrow (A^P)^s V = 0$$
$$\Rightarrow (A^s)^p V= 0.$$

Therefore A^s is a s- nilpotent matrix.

THEOREM 3.1.5

If $A \in C_{nxn}$ is a s-nilpotent matrix then $\bar{A}$ is a s-nilpotent matrix.

PROOF:

We know that A is a s-nilpotent matrix then by definition,

$$A^p V = 0$$

Taking conjugate on both sides,

$$\Rightarrow \overline{(A^pV)} = \bar{0}$$
$$\Rightarrow \overline{A^p}\,\overline{V} = 0$$
$$\Rightarrow (\bar{A})^p V = 0$$

Therefore $\bar{A}$ is a s-nilpotent matrix.

THEOREM 3.1.6

If A and B are s-nilpotent matrices then A+B is a s-nilpotent matrix.

PROOF:

We know that if A and B be s-nilpotent matrices, then by definition,

$$A^pV = 0 \qquad (1)$$

$$B^p V = 0 \qquad (2)$$

Claim: $(A+B)^P V = 0$

$(1) + (2) \Rightarrow A^P V + B^P V = 0$

$\Rightarrow (A+B)^P V = 0$

Therefore A+B is s-nilpotent matrix.

THEOREM 3.1.7

If A and B are s-nilpotent matrix then AB is a s-nilpotent matrix.

PROOF:

We know that if A and B be s-nilpotent matrices, then by definition,

$$A^p V = 0 \qquad (3)$$
$$B^p V = 0 \qquad (4)$$

Claim: $(AB)^P V = 0$

$$(AB)^P V = B^P A^P V$$
$$= B^P (A^P V) = 0$$
$$= B^P .0 = 0$$
$$\Rightarrow (AB)^P V = 0$$

Therefore AB is s-nilpotent matrix.

THEOREM 3.1.8

If $A \in C_{nxn}$ is a s-nilpotent matrix then iA is a s-nilpotent matrix.

PROOF:

We know that A is a s-nilpotent matrix then by definition,

$$A^p V = 0$$

Claim:

$(iA)^P V = 0$

Multiplying both sides by i^P

$$i^P A^P V = 0$$
$$\Rightarrow (iA)^P V = 0$$
$$\Rightarrow (iA)^P V = 0$$

Therefore iA is a s-nilpotent matrix.

THEOREM 3.1.9

If $A \in C_{nxn}$ is a s-nilpotent matrix then –iA is a s-nilpotent matrix

PROOF:

We know that A is a s-nilpotent matrix then by definition.

$$A^p V = 0$$

Claim: $(-iA)^P V = 0$

Multiplying both sides by $-i^P$

$$^{-}i^P A^P V = 0$$
$$\Rightarrow (-iA)^P V = 0$$
$$\Rightarrow (-iA)^P V = 0$$

Therefore -iA is a s-nilpotent matrix.

IV. CONCLUSION

In this paper the concept of s-nilpotent matrices was defined and theorem relating to characterizations of s-nilpotent matrices were derived.

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