Quotient of Ordered Join Hyper Lattices With A Regular Relation

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Abstract- *In this paper, we consider join hyperlattices and we define ordered join hyperlattices. It is already introduced that there exists product of hyperlattices [5]. We introduce the notion of product of two ordered join hyperlattices in this paper. Moreover, we define the quotient of ordered join hyperlattices with a regular relation. Also, we investigate isomorphism on the product of two ordered join hyperlattices with a regular relation.*

Keywords- joinhyper lattice [1], regular relation, Quasi-ordered relation.

I. INTRODUCTION

We define a regular relation on Ordered join hyperlattice such that its quotient [4] is an ordered hyperlattice and we study some properties of such relations.

Definition 1.1:

Let H be a non-empty set. A Hyper operation on H is a map ∘ from H×H to P*(H), the family of non-empty subsets of H. The Couple (H, ∘) is called a hypergroupoid. For any two non-empty subsets A and B of H and x Є H, we define A ∘ B = $\cup_{a\in A, b\in B}$ a ∘ b;

A ∘ x = A ∘ {x} and x ∘B = {x} ∘ B

A Hypergroupoid (H, ∘) is called a Semihypergroup if for all a, b, c of H we have (a ∘ b) ∘ c = a ∘ (b ∘ c). Moreover, if for any element a $\in$ H equalities

A ∘ H = H ∘ a = H holds, then (H, ∘) is called a Hyper group.

Definition 1.2:

A Lattice is a partially ordered set L such that for any two elements x, y of L, glb {x, y} and lub {x, y} exists. If L is a lattice, then we define x ∨ y = glb {x, y} and lub {x, y}.

Definition 1.3:

Let L be a non-empty set, **∧:** L × L → p* (L) be a hyper operation and ∨: L × L →L be an operation. Then (L, ∨, ∧) is a join Hyperlattice if for all x, y, z Є L. The following conditions are satisfied:

1) x Є x ∧ x and x = x∨ x
2) x ∨ (y ∨ z) = (x ∨ y) ∨ z and x ∧ (y ∧ z) = (x ∧ y) ∧ z
3) x ∨ y = y ∨ x and x ∧ y = y ∧ x
4) x Є x ∧ (x ∨ y) ∩ x ∨ (x ∧ y)

Definition 1.4[3]:

Let (L_1, $\vee_1$, $\wedge_1$, $\leq_1$) and (L_2, $\vee_2$, $\wedge_2$, $\leq_2$) be two ordered joinhyperlattice.

Give (L_1× L_2, $\vee'$, $\wedge'$), he two hyperoperations $\vee'$ and ∧' on L_1× L_2 such that for any (x_1, y_1), (x_2, y_2) Є L_1× L_2, we have (x_1, y_1) ∧'(x_2, y_2) = {(u, v); u Є $x_1 \wedge_1 x_2$, v Є $y_1 \wedge_2 y_2$}, (x_1, y_1) ≤ (x_2, y_2) if and only if $x_1 \leq_1 x_2$, $y_1 \leq_2 y_2$.

The Hyper operation $\vee'$ is defined similar to ∧'.

Definition 1.5:

Let $\mathcal{R}$ be an equivalence relation on a non-empty set L and A, B $\subseteq$ L, A $\bar{\bar{\mathcal{R}}}$ B means that for all a Є A, there exists some b Є B such that a $\mathcal{R}$ b,for all b' Є B, there exists a' Є A such that a' $\mathcal{R}$ b'.

Also, $\mathcal{R}$ is called a regular relation respect to ∧ if x $\mathcal{R}$ y implies that x ∧ z $\bar{\bar{\mathcal{R}}}$ y ∧ z, or all x, y, z Є L. $\mathcal{R}$ is called a Regular relation if it is regular respect to ∨ and ∧, at the same time.

Definition 1.6:

An Ideal P of a join hyperlattice L is Prime [2] if for all x, y ∈ L and x ∨ y ∈ P, we have x ∈ P and y ∈ P.

II. QUOTIENT OF ORDERED JOIN HYPERLATTICES WITH A REGULAR RELATION.

In this section, we study special relation which is regular on ordered join hyperlattices which has connection with order on L and we derive ordered join hyperlattice from an ordered join hyperlattice with such regular relation.

Let $(L,\vee,\wedge,\leq)$ be an ordered join hyperlattice and ν be a relation which is transitive and contains the relation $\leq$. Moreover, for any x, y ∈ L, if $x \,\nu\, y$, we have $x \wedge z \,\bar{\nu}\, y \wedge z$ and $x \vee z \,\bar{\nu}\, y \vee z$, for all z ∈ L and $x \in y \wedge z$ implies that $x\nu y$, $y\nu x$, $x\nu z$, $z\nu x$, we call such relations as quasi-ordered relations. We know that if ν is a regular relation, the quotient L/ν is a hyperlattice. But this relaton is not equivalence relation. So, we define $\nu^* = \{(a, b) \in \nu\times\nu;\ a \,\nu\, b,\ b \,\nu\, a\}$.

Theorem 2.1:

Let $(L,\vee,\wedge,\leq)$ be an ordered strong join hyperlattice and ν^* be a relation defined by

$\nu^* = \{(a, b) \in \nu\times\nu;\ a \,\nu\, b,\ b \,\nu\, a\}$. Thus, L/ν^* is an ordered hyperlattice.

Proof:

We can easily show that ν^* is an equivalence relation.

Now we show that ν^* is a regular relation.

Let $x\nu^* y$, $z \in L$ and $x' \in x \wedge z$.

Thus, $x \,\nu\, y$ and $y \,\nu\, x$.

Therefore $x \wedge z \,\bar{\nu}\, y \wedge z$, $y \wedge z \,\bar{\nu}\, x \wedge z$ and we conclude that there exists $y' \in y \wedge z$ such that $x' \,\nu\, y'$.

By the property of ν, we have $y' \,\nu\, z$ and $z \,\nu\, x'$.

So, $y' \,\nu\, x'$ and $x \wedge z \,\bar{\nu}^*\, y \wedge z$.

Since $\vee$ is a binary operation, we show that $x \vee z \,\bar{\nu}^*\, y \vee z$.

So, ν^* is a regular relation and L/ν^* is a hyperlattice.

We have to show that L/ν^* is an ordered hyperlattice.

Let $\nu^*(x) \preccurlyeq \nu^*(y)$.

Since L/ν^* is a hyperlattice, we have $\nu^*(x) \curlywedge \nu^*(z) = \nu^*(z')$ where $z' \in x \wedge z$ and $\nu^*(y) \curlywedge \nu^*(z) = \nu^*(w)$, $w \in y \wedge z$.

Thus, there exists $x' \in \nu^*(x)$ and $y' \in \nu^*(y)$ such that $x' \leq y'$.

Therefore, we have $(x' \wedge z) \leq (y' \wedge z)$ and so $(x' \wedge z) \,\nu\, (y' \wedge z)$.

So, $\nu^*(x') \curlywedge \nu^*(z) \preccurlyeq \nu^*(y') \curlywedge \nu^*(z)$.

Since $\nu^*(x') = \nu^*(x)$ and $\nu^*(y') = \nu^*(y)$, we have $\nu^*(x) \curlywedge \nu^*(z) \preccurlyeq \nu^*(y) \curlywedge \nu^*(z)$.

Therefore, L/ν^* is an ordered hyperlattice.

Theorem 2.2:

Let $(L, \vee, \wedge, \leq)$ be an ordered strong join hyperlattice and ν be a quasi-ordered relation. There is one to one correspondence between quasi-ordered relations on L which contain ν and quasi ordered relations on L/ν^*.

Proof:

Let η be a quasi-ordered relation on L/ν^*.

We have to prove that $\tau = \{(x, y);\ (\nu^*(x), \nu^*(y))\ \eta\}$ is a quasi-ordered relation on L which contains ν.

Let $x \leq y$. So, $x \,\nu\, y$ and $(\nu^*(x), \nu^*(y)) \in L/\nu^*$.

Since η is a quasi-ordered relation, it is clear that $(\nu^*(x), \nu^*(y)) \in \eta$ and so $(x, y) \in \tau$ and $\leq\, \subseteq \tau$.

We can prove that τ has the transitive property.

Now, let $x \in y \wedge z$. Therefore, $\nu^*(x) \in \nu^*(y) \curlywedge \nu^*(z)$ where $\curlywedge$ is a hyper operation on L/ν^*.

Therefore, $(\nu^*(x), \nu^*(y)) \in \eta$ and $(\nu^*(x), \nu^*(z)) \in \eta$.

So, $(x, y) \in \tau$, $(x, z) \in \tau$, $(y, x) \in \tau$, $(z, x) \in \tau$ and let $(x, y) \in \tau$, $a \in x \wedge z$.

So, $\nu^*(a) \in \nu^*(x) \curlywedge \nu^*(z)$ and since η is a quasi-ordered relation, there exists $\nu^*(b) \in \nu^*(y) \curlywedge \nu^*(z)$ such that $(\nu^*(a), \nu^*(b))\ \eta$.

Hence $(a, b)\ \tau$.

Also, we can show for $\vee$ that τ is a quasi-ordered relation on L.

Similarly, if we have a quasi-ordered relation on L which contains ν, then there exists a quasi-ordered relation on L/ν^*.

Theorem 2.3:

Let $(L, \bigvee, \wedge, \leq)$ be an ordered strong join hyperlattice and ν, τ be two quasi-ordered relations on L such that $\nu \subseteq \tau$ and $\nu^*(x)$ τ/ν $\nu^*(y)$ if and only if there exists a $\in \nu^*(x)$, then there exists b $\in \nu^*(y)$, a ν b. Then, τ/ν is a quasi-ordered relation on L/ν^*.

Proof:

Let $(\nu^*(x), \nu^*(y)) \in \tau/\nu$.

Thus, there exists x $\in \tau^*(a)$, y $\in \nu^*(b)$ such that x ν y.

Hence, a ν x and y ν b.

So, a ν b and since $\nu \subseteq \tau$, we have a τ b.

We can easily prove that τ/ν contains $\preccurlyeq$on L/ν^* and has the transitive property.

Now, we let $(\nu^*(x), \nu^*(y)) \in \tau/\nu$ and $\nu^*(z) \in L/\nu^*$, $\nu^*(c) \in \nu^*(x) \curlywedge \nu^*(z)$.

Thus, (x, y) $\in \nu$ and c $\in$ x $\wedge$z.

Since τ is a quasi-ordered relation, there exists u $\in$ y $\wedge$z such that c τ u.

Therefore, there exists $\nu^*(u) \in \nu^*(y) \curlywedge \nu^*(z)$ such that $(\nu^*(c), \nu^*(u)) \in \tau/\nu$.

Also, let $\nu^*(z) \in \nu^*(x) \curlywedge \nu^*(y)$.

Thus, z $\in$ x $\wedge$y and z τ x, z τ y, x τ z, y τ z.

Therefore, $(\nu^*(z), \nu^*(x)) \in \tau/\nu$, $(\nu^*(z), \nu^*(y)) \in \tau/\nu$, $(\nu^*(x), \nu^*(z)) \in \tau/\nu$ and $(\nu^*(y), \nu^*(z)) \in \tau/\nu$.

Similarly, we can prove for $\bigvee$.

In the following theorem we are going to investigate quasi-ordered relation on the product of two ordered join hyperlattices.

Theorem 2.4:

Let $(L_1, \bigvee_1, \wedge_1, \leq_1)$ and $(L_2, \bigvee_2, \wedge_2, \leq_2)$ be two ordered strong join hyperlattices and ν_1, ν_2 be quasi-ordered relations on $L_1$and L_2. Then, $(L_1 \times L_2) / \nu^*$ is isomorphic to $L_1/\nu_1^* \times L_2/\nu_2^*$.

Proof:

We define f: $(L_1 \times L_2) / \nu^* \longrightarrow L_1/\nu_1^* \times L_2/\nu_2^*$ by $f(\nu^*(a), \nu^*(b)) = (\nu_1^*(a), \nu_2^*(b))$.

We can show that f is well defined and one to one.

Now, we claim that f is a homomorphism between two ordered join hyperlattices.

$f(\nu^*(a_1, b_1) \curlywedge \nu^*(a_2, b_2)) = f(\nu^*(u, v))$ where u $\in a_1 \wedge a_2$, v $\in b_1 \wedge b_2$.

So, we have $f(\nu^*(a_1, b_1) \curlywedge \nu^*(a_2, b_2)) = (\nu^*(a_1) \wedge_1 \nu^*(a_2), \nu^*(b_1) \wedge_2 \nu^*(b_2))$

$= f(\nu^*(a_1, b_1)) \times f(\nu^*(a_2, b_2))$.

Similarly, these relations also hold for the binary operation $\curlyvee$ and by the definition of order on$(L_1 \times L_2)$, if $\nu^*(a_1, b_1) \preccurlyeq \nu^*(a_2, b_2)$, we have (a_1, b_1) ν (a_2, b_2).

Therefore, $(a_1, a_2) \in \nu_1$ and $(b_1, b_2) \in \nu_2$.

Thus, $\nu_1^*(a_1) \leq_1 \nu_1^*(a_2)$ and $\nu_2^*(b_1) \leq_2 \nu_2^*(b_2)$.

Therefore, f is an order preserving map and it is clear that f is onto.

So, f is an isomorphism and the proof is completed.

III. CONCLUSION

In this paper, we have successfully derived a ordered join hyperlattice from a ordered join hyperlattice with a regular relation induced in it. We have also investigated quasi-ordered relations on the product of two ordered join hyperlattices.

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