Fuzzy Meet Hyperlattice

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Abstract- *Hyperlattices are the most developing area in the Lattice Theory. In this paper we introduce notion of Fuzzy Theory in meet hyperlattices and also we investigate the properties of Homomorphism of Fuzzy meet hyperlattices.*

Keywords- Hyperlattice, meet hyperlattice, fuzzy hyperlattice, homomorphism

I. INTRODUCTION

Lattices, especially Boolean algebras, arise naturally in logic, and thus, some of the elementary theory of lattices had been worked out earlier. Nonetheless, there is the connection between modern algebra and lattice theory. Lattices are partially ordered sets in which least upper bounds and greatest lower bounds of any two elements exist. Dedekind discovered that this property may be axiomatized by identities. In this paper, we introduce the concept of complete meet hyperlattices, weak homomorphisms in meet hyperlattice and we investigate their properties.

1. PRELIMINARIES

In this section we see some of the basic definitions and conditions, which we use in this paper.

Definition 1.1:

An Algebra (L, $\vee$,$\wedge$) is called a Lattice[4], if L is a non-empty set, $\wedge$ and $\vee$ are binary operations on L, then both $\vee$ and $\wedge$ are

1) Idempotent
2) Commutative, and
3) Associative, and they satisfy the
4) Absorption law.

Proposition 1.2:

Let L be a non-empty set with two binary operations $\wedge$and$\vee$. Let a, b, c $\in$ L, then the following conditions are satisfied:

1) $a \wedge a = a$ and $a \vee a = a$
2) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$
3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$
4) $(a \wedge b) \vee a = a$ and $(a \vee b) \wedge a = a$

Then, (L,$\vee$,$\wedge$) is a Lattice.

Definition 1.3:

Let L be a non-empty set with two hyper operations$\wedge$and$\vee$. The triplet (L,$\vee$, $\wedge$) is called as hyperlattice if the following identities holds for all a, b, c $\in$ L.

1) $a \in a \wedge a$ and $a \in a \vee a$
2) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$
3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ and $(a \vee b) \vee c = a \vee (b \vee c)$
4) $a \in a \wedge (a \vee b)$ and $a \in a \vee (a \wedge b)$

Definition 1.4:

Let (L_1,$\vee_1$, $\wedge_1$) and (L_2, $\vee_2$, $\wedge_2$) be two hyperlattices. A map f: ($L_1 \longrightarrow L_2$) is said to be a

1) weak homomorphism if $f(a \wedge_1 b) \subseteq f(a) \wedge_2 f(b)$ and $f(a \vee_1 b) \subseteq f(a) \vee_2 f(b)$

for all a, b $\in L_1$.

2) Homomorphism if $f(a \wedge_1 b) = f(a) \wedge_2 f(b)$ and $f(a \vee_1 b) = f(a) \vee_2 f(b)$

for all a, b $\in L_1$.

If such a homomorphism f is surjective, then f is called an epimorphism.

If the homomorphism f is injective, then f is called a monomorphism

If f is bijective, then f is called as isomorphism from (L_1, $\vee_1$, $\wedge_1$) to the hyperlattice (L_2, $\vee_2$, $\wedge_2$).

Note 1.5:

Let L be a non-empty set and F*(L) be the set of all non-zero fuzzy subset of L. A map $\circ$ is called as a fuzzy hyper operation on L defined by $\circ$: L × L ⟶ F*(L), where any pair (a, b) of elements of L × L is associated with a non-zero fuzzy subset a $\circ$ b.

If a and b are two non-zero fuzzy subsets of a hyper groupoid (L, $\circ$) for all x, y ∈ L, then we define

1) $(x \circ a)(y) = sup_{t\in L}\{(x \circ t)(y) \cap a(t)\}$, $(a \circ x)(y) = sup_{t\in L}\{a(t) \cap (t \circ x)(y)\}$
2) $(a \circ b)(x) = sup_{t\in L}\{a(p) \cap (p \circ q)(y) \cap b(q)\}$

Remark 1.6:

If A is a non-empty subset of L, then we denote the characteristic function of A by χ_A, where for all y ∈ L, we have

$$\chi_A(y) = \begin{cases} 1, & y \in A \\ 0, & y \in A \end{cases}$$

In particular, for all x, y ∈ L, if A = {x}, then we denote $\chi_{\{x\}}(y) = \chi_b(y)$, which means that

$$\chi_x(y) = \begin{cases} 1, & y = a \\ 0, & y \neq a \end{cases}$$

Definition 1.7:

Let L be a non-empty set with two hyper operation $\vee$ and $\wedge$. The Triplet (L, $\vee$, $\wedge$) is called a fuzzy hyperlattice[1], if the following identities holds for all a, b,c ∈ L.

1) $(a \vee a)(a) > 0$ and $(a \wedge a)(a) > 0$
2) $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$
3) $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
4) $(a \vee (a \wedge b))(a) > 0$ and $(a \wedge (a \vee b))(a) > 0$.

II. RELATION BETWEEN FUZZY MEET HYPERLATTICE AND MEET HYPERLATTICE

In this chapter, we investigate the relation between fuzzy meet hyperlattice and meethyperlattice [2]. We derive prove some of the theorems on them.

Definition 2.1:

Let L be a non-empty set, $\vee$: L × L ⟶ F*(L),be a fuzzy hyper operation and $\bigwedge$: L × L ⟶ L be a operation. Then, (L, $\vee$, $\bigwedge$) is a fuzzy meet hyperlattice if for all x, y, z ∈ L the following conditions holds:

1) $(x \vee x)(x) > 0$ and $(x \bigwedge x)(x) > o$
2) $x \vee y = y \vee x$ and $x \bigwedge y = y \wedge x$
3) $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \bigwedge y) \bigwedge z = x \bigwedge (y \bigwedge z)$
4) $(x \vee (x \bigwedge y))(x) > 0$ and $(x \bigwedge (x \vee y))(x) > 0$.

Example 2.2:

Let (L, $\cap$, $\cup$) be a lattice. If we define the fuzzy hyper operation on L for all x, y ∈ L,

$x \vee y = \chi_{\{x,y\}}$ and the operation

$x \wedge y = \chi_{x \cap y}$, then (L, $\vee$, $\bigwedge$) is a fuzzy meet hyperlattice.

Proof:

1) It is clear that

$(x \vee x)(x) > 0$ and $(x \bigwedge x)(x) > o$ and

2) $x \vee y = y \vee x$ and $x \bigwedge y = y \bigwedge x$ [since x, y ∈ L]
3) Fuzzy associative laws have to be proved, for all x, y, z, a ∈ L

$((x \vee y) \vee z)(a) = sup_{t\in L}\{(x \vee y)(t) \cap (t \vee z)(a)\}$
$= sup_{t\in L}\{\chi_{\{x,y\}}(t) \cap (t \vee z)(a)\}$

$$= (x \vee z)(a) \cup (y \vee z)(a)$$
$$= \chi_{\{x,z\}}(a) \cup \chi_{\{y,z\}}(a)$$
$$= \chi_{\{x,y,z\}}(a)$$

and

$$x \vee (y \vee z)(a) = sup_{t\in L}\{(x \vee t)(a) \cap (y \vee z)(t)\}$$
$$= sup_{t\in L}\{(x \vee t)(a) \cap \chi_{\{y,z\}}(t)\}$$
$$= (x \vee y)(a) \cup (x \vee z)(a)$$
$$= \chi_{\{x,y\}}(a) \cup \chi_{\{x,z\}}(a)$$
$$= \chi_{\{x,y,z\}}(a)$$

Hence $(x \vee y) \vee z = x \vee (y \vee z)$.

Similarly, $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ can be proved.

4) Fuzzy absorption law:

for all x, y, z $\in$ L and a $\in$ L,

$$(x \vee (x \wedge y))(x) = sup_{t\in L}\{(x \vee t)(x) \cap (x \wedge y)(t)\}$$
$$= sup_{t\in L}\{(x \vee t)(x) \cap \chi_{\{x\cap y\}}(t)\}$$
$$= (x \vee (x \cap y))(x)$$
$$= \chi_{\{x, x\cap y\}}(x) = 1 > 0$$

$$(x \wedge (x \vee y))(x) = sup_{t\in L}\{(x \wedge t)(x) \cap (x \vee y)(t)\}$$
$$= sup_{t\in L}\{(x \wedge t)(x) \cap \chi_{\{x,y\}}(t)\}$$
$$= (x \wedge x)(x) \cup (x \wedge y)(x)$$
$$\geq (x \wedge x)(x) > 0$$

The proof is complete.

Example 2.3:

Let (L, $\cap$,$\cup$) be a lattice. If we define the fuzzy hyper operation $\vee$ on L for all x, y $\in$ L, as

$x \vee y = \chi_{\{x,y\}}$ and the operation $\wedge$ is defined as for all a $\in$ L,

$$(x \wedge y)(a) = \begin{cases} \{½\}, & a = (x \cap y) \\ 0, & \text{otherwise} \end{cases}$$

Then (L,$\vee$, $\wedge$) is a fuzzy meet hyperlattice.

Example 2.4:

Let (L,$\cap$,$\cup$) be a lattice. If we define the fuzzy hyper operation $\vee$ on L for all x, y $\in$ L, as

$x \vee y = \chi_{\{x\cup y\}}$ and the operation $\wedge$ is defined as for all a $\in$ L,

$$(x \wedge y)(a) = \begin{cases} \{½\}, & a = (x \cap y) \\ 0, & \text{otherwise} \end{cases}$$

Then (L, $\vee$, $\wedge$) is a fuzzy meet hyperlattice.

Definition 2.5:

Let L be a non-empty set, $\vee$**:** L × L → F* (L) be a hyper operation and $\wedge$: L × L →L be an operation. Then (L,$\vee$, $\wedge$) is a meethyperlattice[3] if for all x, y, z Є L. The following conditions are satisfied:

1) x Є $x \vee x$ and $x = x \wedge x$
2) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$
3) $x \vee (y \vee z) = (x \vee y) \vee z$ and $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
4) x Є $x \wedge (x \vee y) \cap x \vee (x \wedge y)$

On considering a fuzzy meet hyperlattice (L,$\vee$, $\wedge$) and defining the hyper operations on L for all x, y Є L,

$x \otimes y$ = {aЄ L | $(x \vee y)(a) > 0$ } and

$x \oplus y$ = {aЄ L | $(x \wedge y)(a) > 0$}, then we obtain a hyperlattice.

Theorem 2.6:

If (L,$\vee$, $\wedge$) is a fuzzy meet hyperlattice, then (L, $\otimes$, $\oplus$) is a meet hyperlattice, which is called the associated meet hyperlattice.

Proof:

1) clearly, x Є $x \otimes x$ and x Є $x \oplus x$

2) x $\otimes$ y = y $\otimes$ x and x $\oplus$ y = y $\oplus$ x
3) Associative laws:

We claim that x $\otimes$ (y $\otimes$ z) = (x $\otimes$ y) $\otimes$ z, for all x, y, z Є L, then for all a Є L,

if a Є x $\otimes$ (y $\otimes$ z),

then there exists b Є y $\otimes$ z, such that a Є x $\otimes$ b which gives (y $\vee$ z) (b) > 0 and (x $\vee$ b) > 0

Hence, x $\vee$ (y $\vee$ z) (a) = $sup_{t\in L}$\{(x $\vee$t) (a) $\cap$ (y $\vee$z) (t)\}
≥ (x $\vee$b) (a) $\cap$ (y $\vee$z) (b) > 0
Since x $\vee$ (y $\vee$ z) = (x $\vee$ y) $\vee$z, it follows that ((x $\vee$ y) $\vee$z) (a) > 0

which implies that
$sup_{t\in L}$\{(x $\vee$y) (t) $\cap$ (t $\vee$z) (a)\} > 0
Hence there is t' Є L, such that
(x $\vee$ y) (t') > 0 and (t' $\vee$ z) (a) > 0.
This shows that t' Є x $\otimes$ y and a Є t' $\otimes$ z.
Therefore, we get a Є (x $\otimes$ y) $\otimes$ z
Hence, x $\otimes$ (y $\otimes$ z) $\subseteq$(x $\otimes$ y) $\otimes$ z.
Conversely, (x $\otimes$ y) $\otimes$ z $\subseteq$ x $\otimes$ (y $\otimes$ z).
So, we arrive at the equality,
x $\otimes$ (y $\otimes$ z) = (x $\otimes$ y) $\otimes$ z.
Similarly, we can show that,
x $\oplus$ (y $\oplus$ z) = (x $\oplus$ y) $\oplus$ z.

4) Absorption Law:

(x $\vee$ (x $\wedge$ y)) (x) = $sup_{t\in L}$\{(x $\vee$t) (x) $\cap$ (x $\wedge$ y) (t)\} > 0, then there exists t' Є L such that,
(x $\vee$ y) (t') > 0 and (x $\vee$t') (x) > 0 which shows that there exists t' Є x $\oplus$ y,
such that x Є x $\otimes$ t'.
Therefore, x Є x $\otimes$(x $\oplus$ y).
Similarly, we get x Є x $\oplus$(x $\otimes$ y).
Combining 1), 2), 3) and 4) we have that (L, $\otimes$, $\oplus$) is a meet hyperlattice.

Remark 2.7:

Now, on considering a meet hyperlattice (L, $\otimes$, $\oplus$),and defining the hyperoperation on L:
for all x, y Є L,

x $\vee$y = $\chi_{x\otimes y}$

x $\wedge$ y = $\chi_{x\oplus y}$, then we obtain a fuzzy meet hyperlattice.

Theorem2.8:

If (L,$\otimes$, $\oplus$), is a meet hyperlattice, then (L, $\vee$, $\wedge$) is a fuzzy meethyperlattice, which is called the associated fuzzy meet hyperlattice.

Proof:

1) (x $\vee$x) (x) > 0 and (x $\wedge$ x)(x) > 0 is clear and

2) x $\vee$ y = y$\vee$ x and x $\wedge$ y = y $\wedge$ x also holds
3) Fuzzy associative law:

For all x,y, z, a Є L, we have

((x $\vee$y) $\vee$z) (a)=$sup_{t\in L}$\{(x $\vee$y) (t) $\cap$ (t $\vee$z) (a)\}
= $sup_{t\in L}$ \{ $\chi_{x\otimes y}$ (t) $\cap$ $\chi_{t\otimes z}$ (a)\}
= 1, if a Є (x$\otimes$ y) $\otimes$ z
0, otherwise
Then,

(x$\vee$(y$\vee$z) (a) =$sup_{t\in L}$\{(x $\vee$t) (a) $\cap$ (y $\vee$z) (t)\}
= $sup_{t\in L}$ \{ $\chi_{x\otimes t}$ (a) $\cap$ $\chi_{y\otimes c}$ (t)\}
= 1, if a Є (x$\otimes$ y) $\otimes$ z
0, otherwise
We get,
(x $\vee$ y) $\vee$ z = x$\vee$(y $\vee$ z) [since, x$\otimes$ (y $\otimes$ z) = (x $\otimes$y) $\otimes$z]
Similarly, we can prove that

(x$\wedge$y)$\wedge$z = x $\wedge$ (y $\wedge$z)
4) Fuzzy Absorption law,

For all x, y Є L we have, (x $\vee$(x $\wedge$ y)) (x) = $sup_{t\in L}$ \{(x $\vee$t) (x) $\cap$ (x $\wedge$ y)(t)\}
= $sup_{t\in L}$ \{ $\chi_{x\otimes t}$ (x) $\cap$ $\chi_{x\otimes y}$ (t)\}
= $\chi_{x\otimes(x\oplus y)}$(x)
= 1 > 0

Hence the proof is completed.

From the above two theorems we have the following facts.

Denote the class of all fuzzy meet hyperlattice by fuzzy meet hyper lattice and the class of all meet hyperlattice by meet hyper lattice. We can form two maps

1) θ:FJHL → JHL, θ ((L, $\vee$, $\wedge$)) = (L, $\otimes$, $\oplus$), where for any x, y Є L, we have
x $\otimes$ y = {a Є L | (x $\vee$y) (a)> 0} and

2) π: JHL → FJHL, π((L, $\otimes$, $\oplus$)) = (L, $\vee$, $\wedge$), where for any x, y Є L, we have
(x $\vee$y) = $\chi_{\{x \otimes y\}}$ and (x $\vee$y) = $\chi_{\{x \oplus y\}}$

III. HOMOMORPHISM OF FUZZY MEET HYPERLATTICE AND MEET HYPERLATTICE

In this section we introduce the notion of homomorphism of fuzzy meet hyperlattices and we study the connections between meet hyperlattices homomorphism and meet hyperlattice homomorphism.

Definition 3.1:

Let (L_1, $\vee_1$, $\wedge_1$) and (L_2, $\vee_2$, $\wedge_2$) be two fuzzy meet hyperlattices. A map f: ($L_1 \longrightarrow L_2$) is said to be

1) Weak homomorphism if f (x $\vee_1$ y) $\subseteq$ f(x)$\vee_2$, f(y) and f(x$\wedge_1$ y) $\subseteq$ f(a)$\wedge_2$ f(b) for all x, y $\in L_1$.
2) Homomorphism if f (x $\vee_1$ y) = f(x)$\vee_2$ f(y) and f(x $\wedge_1$y) = f(a)$\wedge_2$ f(b)

for all x, y $\in L_1$.

If such a homomorphism is said to be surjective, injective or bijective then the mapping f is called as epimorphism, monomorphism or isomorphism from the meet hyperlattice (L_1,$\vee_1$,$\wedge_1$) to the meet hyperlattice (L_2, $\vee_2$, $\wedge_2$) respectively.

Theorem 3.2:

Let (L_1,$\vee_1$, $\wedge_1$) and (L_2,$\vee_2$,$\wedge_2$) be two fuzzy meet hyperlattices and (L_1,$\otimes_1$, $\oplus_1$) = θ ((L_1,$\vee_1$, $\wedge_1$)) and (L_2,$\otimes_2$,$\oplus_2$) = θ ((L_2,$\vee_2$,$\wedge_2$)) be their associated meet hyperlattices , respectively. If f: $L_1 \to L_2$ is a weak homomorphism of fuzzy meethyperlattices, then f is a weak homomorphism of the associated meethyperlattices, too.

Proof:

Since f: $L_1 \to L_2$ is a weak homomorphism of fuzzy meet hyperlattices, we have
f (x$\vee_1$y) ⊆ f(x)$\vee_2$f(y)and f(x$\wedge_1$y))⊆ f(x)$\wedge_2$f(y) for all x,yϵL_1.
This shows that

(f(x$\vee_1$y))(b) ≤ (f(x)$\vee_2$f(y))(b) and
(f(x$\wedge_1$y))(b) ≤ (f(x)$\wedge_2$f(y))(b) for all bϵL_2.

Let aϵx$\otimes_1$y, which means that (x$\vee_1$y ((a)>0 and let b = f(a)
Then, (f(x$\vee_1$y)) (b) = sup{(x$\vee_1$y) (t) |f (t) = b, t ϵL_1}
≥(x$\vee_1$y)(a) >0
It shows that (f(x)$\vee_2$f(y)(b)>0.
Hence b = f (a) ϵ f(x)$\otimes_2$ f(y).
Therefore we get, f(x $\otimes_1$ y) ⊆ f(x) $\otimes_2$ f(y).
Similarly, we can show that

f(x $\oplus_1$ y) ⊆ f(x) $\oplus_2$f(y).

Therefore, f is a weak homomorphism between the associated meet hyperlattices (L_1,$\otimes_1$, $\oplus_1$) and (L_2,$\otimes_2$, $\oplus_2$).

IV. CONCLUSION

Hence, we have successfully introduced the fuzzy meet hyperlattice. And we investigated some of their properties.

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