The Similarity Solution of The Imbibition Phenomenon of Immiscible Fluids In Porous Media

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Abstract- *The Imbibition phenomenon of miscible fluids in porous media is discussed by regarding the cross-sectional flow velocity as time dependent in a specific form. The mathematical formulation of the phenomenon yields a non-linear partial differential equation. This partial differential equation is transformed into ordinary differential equation by using infinitesimal transformations group technique of similarity analysis. An analytical solution of the later is derived in terms of error function. The solution obtained is physically consistent with the results of earlier researchers and which is also more classical than other results obtained by various researchers. This type of phenomenon has been of great concern to hydrologists who have been studying the problem of displacement of fresh water by sea water in coastal areas. The oil industry has also become involved in miscible displacement studies in connection with the possibility of flushing oil by solvents from reservoirs.*

Keywords- Similarity solution, Infinitesimal Transformations, Porous media, Imbibition phenomenon.

I. INTRODUCTION

This paper discusses mathematically the phenomenon of imbibition in double phase flow of two immiscible fluids in a homogeneous porous medium with capillary pressure. The imbibition phenomenon arises due to the difference in the wetting abilities of the phases. It is assumed that the injection of a preferentially wetting fluid into a porous medium saturated with another non-wetting fluid is initiated by imbibition and the injected fluid phases of different wetting abilities with respect to the porous media.

By assuming the validity of the Darcy's law, a mathematical formulation of the imbibition phenomenon yields a non-linear partial differential equation which has been reduced into the well-known Abel's equation of the second kind of using Birkhoff's technique of one parameter group transformation. An analytical solution of the later has been obtained in terms of transcendental functions.

If a porous medium filled with some fluid is brought in contact with another fluid which preferentially wets the medium then there is a spontaneous flow of the wetting fluid into the medium, and a counter flow of the resident fluid from the medium. The phenomenon arises due to the difference in the wetting abilities of the fluids, is called countercurrent imbibition [1]. This phenomenon has been formally discussed for a homogeneous porous medium b many researchers, for example, Graham and Richardson [2], Scheidegger [3]. Bokserman et al [4] has described the physics of oil-water motion in a cracked porous medium. Verma [5, 6] has employed perturbation procedure and similarity methods to obtain explicit analytical solutions of imbibition equation in a heterogeneous cracked porous medium. Mehta and Verma [7] have discussed analytically the phenomenon of imbibition in porous media, under certain condition, by using a singular perturbation approach. Verma and Ram Mohan [8] have discussed the existence and uniqueness of similarity solutions of imbibition equation. A theoretical model for imbibition phenomenon of groundwater replenishment in a double-phase flow through a porous medium is presented by Mishra et al [9]. A general review for double-phase immiscible fluids flow in fractured porous media was given by Bear and Braester [10]. Pradhan and Verma [11] have obtained the numerical solution of a specific imbibition phenomenon by using Crank-Nicolson Scheme for finite differences. Exponential self-similar solution for imbibition phenomenon in porous media with capillary pressure is derived by Meher et. al. [12].

In this paper, we have derived the governing equation and discussed the similarity solution of the imbibition phenomenon arising in the flow of two immiscible fluids flow through porous media with the effect of capillary pressure by using infinitesimal transformations group technique of similarity analysis. Under standard assumptions a classical form of the displaced fluid saturation has been obtained. Thus, we have established a new relationship between capillary pressure and displacing fluid saturation.

II. STATEMENT OF PROBLEM

We consider here that a cylindrical piece of homogeneous porous medium of length L is fully saturated with a non-wetting fluid n (e.g. oil). It is assumed that three sides of the porous medium are bounded by impermeable surfaces and one open end is exposed to an adjacent formation of a wetting fluid w (e.g.water) which the medium preferentially relative to n. Such circumstances give rise to the phenomenon of one-dimensional counter flow of the non-wetting fluid from the medium.

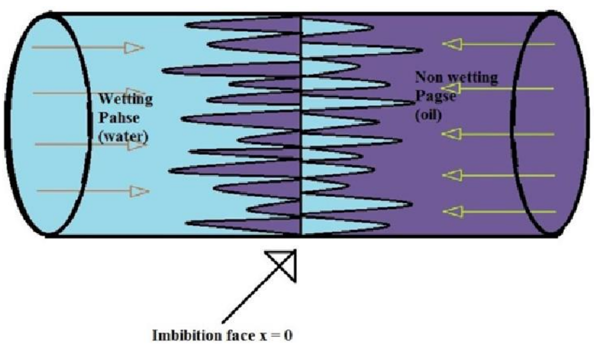

Fig. 1 Imbibition in horizontal cylindrical homogeneous porous matrix.

III. MATHEMATICAL FORMULATION

By assuming the validity of Darcy's law for the phase flow system, we may write the seepage velocities of the wetting phase (v_w) and the non-wetting phase (v_n) as [13]:

$$v_w = -\frac{k_w}{\delta_w} K \frac{\partial p_w}{\partial x} \quad (1)$$

$$v_n = -\frac{k_n}{\delta_n} K \frac{\partial p_n}{\partial x} \quad (2)$$

where k_w and k_n are the relative permeability's; p_w and p_n are pressures; δ_w and δ_n are the constant kinematic viscosities of the wetting phase and non-wetting phase respectively and K is the permeability of the homogeneous porous medium. The co-ordinate x is measured along the axis of the cylindrical medium, the origin being located at the imbibition face $x = 0$

Since $v_w = -v_n$, (3)

For the imbibition phenomenon [3], therefore, from equations (1) and (2), we may write

$$\frac{k_w}{\delta_w}\frac{\partial p_w}{\partial x} + \frac{k_n}{\delta_n}\frac{\partial p_n}{\partial x} = 0 \quad (4)$$

The definition of capillary pressure p_c [3] gives

$$p_c = p_n - p_w \quad (5)$$

Combining equations (4) and (5), we get

$$\left(\frac{k_w}{\delta_w} + \frac{k_n}{\delta_n}\right)\frac{\partial p_w}{\partial x} + \frac{k_n}{\delta_n}\frac{\partial p_c}{\partial x} = 0 \quad (6)$$

Substituting the value of $\frac{\partial p_w}{\partial x}$ from the equation (6) into the equation (1), we obtain

$$v_w = K \frac{\frac{k_w}{\delta_w}\frac{k_n}{\delta_n}\frac{\partial p_c}{\partial x}}{\frac{k_w}{\delta_w} + \frac{k_n}{\delta_n}} \quad (7)$$

The equation of continuity for the wetting phase is given by

$$\phi \frac{\partial s_w}{\partial t} + \frac{\partial v_w}{\partial x} = 0 \quad (8)$$

where ϕ is the porosity of the medium and s_w is saturation of the wetting phase.

Equation (8) with the value of v_w from equation (7), Becomes

$$\phi \frac{\partial s_w}{\partial t} + \frac{\partial}{\partial x}\left[\frac{k_w k_n}{k_w \delta_n + k_n \delta_w} K \frac{\partial p_c}{\partial s_w}\frac{\partial s_w}{\partial x}\right] = 0 \quad (9)$$

This equation (9) is a non-linear partial differential equation, which describes the linear countercurrent imbibition phenomenon of two immiscible fluids flow through homogeneous porous cylindrical medium with impervious bounding surfaces on three sides.

As we already know that fictitious relative permeability is the function of displacing fluid saturation. Then at this stage, for definiteness of the mathematical analysis, we assume standard forms for the analytical relationship between the relative permeability, phase saturation and capillary pressure phase saturation [14] as

$$k_w = s_w \quad (10)$$

$$k_n = 1-\alpha S_w \ (11) \text{ and } P_c = \beta S_w (\text{where } \alpha=1.11) \quad (12)$$

Since the present investigation involves water and viscous oil, therefore, according to Scheidegger [3], we have

$$\frac{k_w k_n}{k_w \delta_n + k_n \delta_w} = \frac{k_n}{\delta_n} \quad (13)$$

Substituting the values of k_n and P_c from equations (11) and (12) into the equation (9) and using (13), we get

$$\phi \frac{\partial s_w}{\partial t} + \frac{\partial}{\partial x}\left[\beta \frac{k_n}{\delta_n}\overline{S_w}\frac{\partial \overline{S_w}}{\partial x}\right] = 0 \quad (14)$$

Where

$$\overline{S_w} = 1-\alpha S_w \quad (15)$$

This is the desired non-linear partial differential equation describing the linear countercurrent imbibition phenomenon with capillary pressure.

A set of boundary conditions for the problem may be written as

$$S_w(0,t) = S_{wo}; (t>0) \quad (16)$$

$$S_w(L,t) = S_{wL}; (t>0) \quad (17)$$

where S_{wo} and S_{wL} are saturations at the imbibition face $x=0$ and at the end $x=L$ respectively.

III. SIMILARITY SOLUTION

We seek a one-parameter group of infinitesimal transformation, which takes the (x,t,s)-space into itself and under which equation (14) is invariant:

$$G = \begin{Bmatrix} \bar{x} = x + \varepsilon X \\ \bar{t} = t + \varepsilon T \\ \bar{s} = s + \varepsilon S \end{Bmatrix} \quad (18)$$

The equation (14) can be written as

$$\frac{\partial \overline{S}_w}{\partial T} + \left(\frac{\partial \overline{S}_w}{\partial X}\right)^2 + \overline{S}_w \frac{\partial^2 \overline{S}_w}{\partial X^2} = 0$$
$$\overline{S}_w(0,T) = 1 - S_{wo}; (T>0)$$
$$\overline{S}_w(L,t) = S_{wL}; (T>0) \quad (19)$$

where generators X, T and S_w are functions of x, t and s.

Applying transformations (18) into the equation (19)we get the group G of infinitesimal transformation explicitly as,

$$G = \begin{cases} X = \dfrac{a_0}{2}x + k \\ T = a_0 t + a_1 \\ S_w = s^2 + ks + a_2 \end{cases} \quad (20)$$

Thus, the characteristic equations are

$$\frac{dX}{\frac{a_0}{2}x + k} = \frac{dT}{a_0 t + a_1} = \frac{dS_w}{s^2 + ks + a_2} \quad (21)$$

Now, assume that, $a_0 = 2$ and $k = 1, a_1 = a_2 = 0$, we have,

$$\frac{dX}{x+1} = \frac{dT}{2t} = \frac{dS_w}{s^2+s} \quad (22)$$

From these, we can have the similarity variable $\eta = \dfrac{x+1}{\sqrt{t}}$

and also we have, $S_w = \phi(\eta)$, where $\eta = \dfrac{x+1}{\sqrt{t}}$ (23)Using equations (23) in to the equation (19) we obtain the following ordinary differential equation,

$$2\phi(\eta)\phi''(\eta) + 2\left(\phi^{'}(\eta)\right)^2 - \eta\phi^{'}(\eta) = 0 \quad (24)$$

Equation (24) is the second order nonlinear ordinary differential equation of imbibition phenomenon with capillary pressure in porous media.

Substituting, $\phi(\eta) = \eta^2 u(z), z = ln(\eta), u^{/}(z) = q$ as in [15] into the equation (24), we get

$$uq\frac{dq}{du} + q^2 + (7u-1)q + (6u-1) = 0 \quad (25)$$

It is the Abel's equation of the second kind.

Substitute, $uq = \ln v(u)$ and $\ln v(u) = w(z)$
Into the equation (25) to get,

$$\frac{dw}{dz} + (7u - 1)\frac{1}{u}w + (6u - 1)u = 0 \quad (26)$$

This is the linear differential equation whose solution is given by

$$Wexp\left[\int \frac{7\phi(\eta)-\eta^2}{\eta\phi(\eta)}d\eta\right] = \int\left[\frac{\phi(\eta)}{\eta^4}\left(\eta^2 - 6\phi(\eta)\right).exp\left\{\int \frac{7\phi(\eta)-\eta^2}{\eta\phi(\eta)}d\eta\right\}\right]d\eta + c \quad (27)$$

where c is a constant of integration which is determined considering $7\phi(\eta) = \eta^2$ as equal to zero.

Then the equation (27) takes the form

$$Wexp\left[\int \frac{7\phi(\eta)-\eta^2}{\eta\phi(\eta)}d\eta\right] = \int\left[\frac{\phi(\eta)}{\eta^4}\left(\eta^2 - 6\phi(\eta)\right).exp\left\{\int \frac{7\phi(\eta)-\eta^2}{\eta\phi(\eta)}d\eta\right\}\right]d\eta \quad (28)$$

The above equation (28) represents the solution of the problem under investigation in terms of transcendental functions.

IV. CONCLUSIONS

We have obtained the solution in terms of transcendental functionsof the specific problem of the imbibition phenomenon in a homogeneous porous medium with capillary pressure by using infinitesimal group transformation technique of similarity analysis. We have not included any numerical as well as graphical illustrations due to our particular interest in the mathematical analysis. It is believed that the present similarity solution will provide useful theoretical information of at least one complicated case of imbibition phenomenon of two immiscible liquid flow problems.

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