Further Study on Supra pre-connected sets in Topological Spaces

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Abstract- *O.R.Sayed introduced supra pre-open sets and supra pre-continuous functions in topological spaces. We introduced supra pre-separated sets and supra pre-connected sets in supra topological spaces and studied some of their basic properties. In this paper we investigate some more properties of supra pre-connected sets in supra topological spaces.*

Keywords- Supra open sets, supra pre-open sets, supra pre-separated sets, supra pre-connected sets.

I. INTRODUCTION

The concept of supra topology is fundamental with respect to the investigation of general topological spaces. Also, generalization of openness, such as supra open, supra α-open, supra pre-open, supra b-open, etc. are important in topological spaces. In 1983, A.S.Mashhour, A.A.Allam, F.S.Mahmoud and F.H. Khedr introduced supra topological spaces. Also in 2008, R. Devi, S. Sampathkumar and M. Caldas introduced and studied supra α-open sets. O.R. Sayed introduced supra pre-open sets and supra pre-continuous functions in topological spaces. In 2019, we introduced supra pre-separated sets and supra pre-connected sets in supra topological space and we also studied some of their basic properties. In this paper, we studied some properties of supra pre-connected sets.

Section 2 deals with the preliminary concepts. In section 3, some properties of supra pre-connected sets are discussed.

II. PRELIMINARIES

Throughout this paper (X, μ) (or simply X) denotes a supra topological space. The following definitions are very useful in the subsequent sections.

Definition 2.1: [2] Let X be a non-empty set. The subcollection $\mu \subseteq P(X)$, where P(X) is a power set of X is called a supra topology on X if $\emptyset, X \in \mu$ and μ is closed under the arbitrary union. The pair (X, μ) is called a supra topological space. The members of μ are called supra open sets. The complement of supra open sets are called supra closed sets.

Definition 2.2: [2] The supra interior of a set A, denoted by $\mathrm{Int}^{\mu}(A)$, is the union of the supra open sets included in A. The supra closure of a set A, denoted by $\mathrm{Cl}^{\mu}(A)$, is the intersection of the supra closed sets including A.

Definition 2.3: A subset A of X is called

i. a supra α-open set [1] if $A \subseteq \mathrm{Int}^{\mu}(\mathrm{Cl}^{\mu}(\mathrm{Int}^{\mu}(A)))$
ii. a supra b-open set [5]if $A \subseteq \mathrm{Cl}^{\mu}(\mathrm{Int}^{\mu}(A)) \cup \mathrm{Int}^{\mu}(\mathrm{Cl}^{\mu}(A))$

Definition 2.4: [4] A subset A of X is called a supra pre-open set if $A \subseteq \mathrm{Int}^{\mu}(\mathrm{Cl}^{\mu}(A))$ and the complement of a supra pre-open set is said to be supra pre-closed.

Definition 2.5: [4] The supra pre-interior of a set A, denoted by $\mathrm{Int}_p^{\mu}(A)$, is the union of the supra pre-open sets included in A. The supra pre-closure of a set A, denoted by $\mathrm{Cl}_p^{\mu}(A)$, is the intersection of the supra pre-closed sets including A.

Lemma 2.6: [4] (i) $\mathrm{Int}_p^{\mu}(A) \subseteq A$
(ii) $A = \mathrm{Int}_p^{\mu}(A)$ if and only if A is a supra pre-open set.

Lemma 2.7: [4] (i) $A \subseteq \mathrm{Cl}_p^{\mu}(A)$
(ii) $A = \mathrm{Cl}_p^{\mu}(A)$ if and only if A is a supra pre-closed set.

Lemma 2.8 : [4] $X - \mathrm{Int}_p^{\mu}(A) = \mathrm{Cl}_p^{\mu}(X - A)$ and $X - \mathrm{Cl}_p^{\mu}(A) = \mathrm{Int}_p^{\mu}(X - A)$.

Lemma 2.9: [4] If $A \subseteq B$, then $\mathrm{Cl}_p^{\mu}(A) \subseteq \mathrm{Cl}_p^{\mu}(B)$ and $\mathrm{Int}_p^{\mu}(A) \subseteq \mathrm{Int}_p^{\mu}(B)$.

Definition 2.10: [3] Let (X, μ) be a supra topological space. Let A and B be non-empty subsets of X. Then A and B are said to be supra pre-separated sets if $A \cap \mathrm{Cl}_p^{\mu}(B) = \emptyset$ and $\mathrm{Cl}_p^{\mu}(A) \cap B = \emptyset$.

Definition 2.11: [3] A subset A of X is said to be supra pre-connected if it cannot be represented as a union of two non-empty supra pre-separated sets. If X is supra pre-connected, then X is called a supra pre-connected space.

Lemma 2.12: [3] Let C and D be non-empty subsets of the supra pre-separated sets A and B, respectively. Then C and D are also supra pre-separated.

Lemma 2.13: [3] A non-empty subset C of X is supra pre-connected if and only if for every pair of supra pre-separated sets A and B in X with C ⊆ A ∪ B, one of the following conditions holds:
(a) C ⊆ A and C ∩ B = ∅.
(b) C ⊆ B and C ∩ A = ∅.

III. FURTHER STUDY ON SUPRA PRE-CONNECTED SETS

In this section we discuss some properties and characterizations of supra pre-connected sets.

Theorem 3.1: For a subset G of X, the following conditions are equivalent:

(1) G is supra pre-connected.
(2) There do not exist two supra pre-closed sets A and B such that A ∩ G≠ ∅, B ∩ G≠ ∅, G ⊆ A ∪ B and A ∩ B ∩ G = ∅.
(3) There do not exist two supra pre-closed sets A and B such that G ⊈ A, G ⊈ B, G ⊆ A ∪ B and A ∩ B ∩ G = ∅.

Proof: Let (X, μ) be a supra topological space.
(1) ⇒ (2).
Suppose G is supra pre-connected and there exist two supra pre-closed sets A and B such that A ∩ G≠ ∅, B ∩ G≠ ∅, G ⊆ A ∪ B and A ∩ B ∩ G = ∅. Then (A ∩ G) ∪ (B ∩ G) = (A ∪ B) ∩ G = G. Now, Cl_p^μ(A ∩ G) ∩ (B ∩ G) ⊆ Cl_p^μ(A) ∩ (B ∩ G) = A ∩ B ∩ G = ∅. Similarly, we can prove that (A ∩ G) ∩ Cl_p^μ(B ∩ G) = ∅. This shows that G is not supra pre-connected, which contradicts the hypothesis. Hence (2) holds.
(2) ⇒ (3).
Suppose that there exist supra pre-closed sets A and B such that G ⊈ A, G ⊈ B, G ⊆ A ∪ B and A ∩ B ∩ G = ∅. Then A ∩ G≠ ∅, B ∩ G≠ ∅, which is a contradiction to our assumption. Hence (3) holds.
(3) ⇒ (1).
Suppose that the condition (3) satisfied and G is not supra pre-connected. Then there exist two non-empty subsets C and D such that C ∪ D = G and Cl_p^μ(C) ∩ D = C ∩ Cl_p^μ(D) = ∅. Assume that A = Cl_p^μ(C) and B = Cl_p^μ(D). Hence G ⊆ A ∪ B. Now, A ∩ B ∩ G = Cl_p^μ(C) ∩ Cl_p^μ(D) ∩ (C ∪ D) = (Cl_p^μ(C) ∩ Cl_p^μ(D) ∩ C) ∪ (Cl_p^μ(C) ∩ Cl_p^μ(D) ∩ D) = (Cl_p^μ(D) ∩ C) ∪ (Cl_p^μ(C) ∩ D) = ∅. Thus A ∩ B ∩ G = ∅. Now we prove that, G ⊈ A, G ⊈ B. If G ⊆ A, then Cl_p^μ(D) ∩ G = B ∩ G = B ∩ (G ∩ A) = ∅, which is a contradiction. Thus G ⊈ A. Similarly, we have G ⊈ B. This contradicts (3). Therefore, G is supra pre-connected.

Theorem 3.2: For a subset G of X, the following conditions are equivalent:
(1) G is supra pre-connected.
(2) For any supra pre-separated sets A and B with G ⊆ A ∪ B, we have G ∩ A = ∅ or G ∩ B = ∅.
(3) For any supra pre-separated sets A and B with G ⊆ A ∪ B, we have G ⊆ A or G ⊆ B.

Proof: Let (X, μ) be a supra topological space.
(1) ⇒ (2).
Let G be supra pre-connected. Suppose that A and B are supra pre-separated sets such that G ⊆ A ∪ B, then by lemma 2.12, we have G ∩ A and G ∩ B are supra pre-separated. Since G is supra pre-connected and G = G ∩ (A ∪ B) = (G ∩ A) ∪ (G ∩ B), we have either G ∩ A = ∅ or G ∩ B = ∅.
(2) ⇒ (3).
Assume that (2) holds. If G ∩ A = ∅, then G = G ∩ (A ∪ B) = (G ∩ A) ∪ (G ∩ B) = ∅ ∪ (G ∩ B) = G ∩ B. Hence G = G ∩ B. This implies that, G ⊆ B. Similarly, if G ∩ B = ∅, then G ⊆ A. Hence (3) holds.
(3) ⇒ (1).
Let the condition (3) be satisfied. Suppose A and B are supra pre-separated and G = A ∪ B, then by (3), either G ⊆ A or G ⊆ B. If G ⊆ A, then B = B ∩ G ⊆ B ∩ A ⊆ B ∩ Cl_p^μ(A) = ∅. Therefore, if G ⊆ A, then B = ∅. Similarly, if G ⊆ B, then A = ∅. Hence g cannot be represented as a union of two non-empty supra pre-separated sets. Therefore, G is supra pre-connected.

Theorem 3.3: Let G be a supra pre-connected subset of X. If G ⊆ H ⊆ Cl_p^μ(G), then H is also supra pre-connected.

Proof: Let (X, μ) be a supra topological space. Suppose H is not supra pre-connected, then by Theorem 3.1, there exist two supra pre-closed sets A and B such that H ⊈ A, H ⊈ B, H ⊆ A ∪ B and A ∩ B ∩ H = ∅. Since G ⊆ H, we have G ⊆ A ∪ B and A ∩ B ∩ G = ∅. Now, we prove that G ⊈ A, G ⊈ B. If G ⊆ A, then Cl_p^μ(G) ⊆ Cl_p^μ(A) = A. Since Cl_p^μ(G) ⊆ A, H ⊆ A, which is a contradiction to H ⊈ A. Hence G ⊈ A. Similarly, we can prove that G ⊈ B. This contradicts the fact that G is supra pre-connected. Hence H is supra pre-connected.

Theorem 3.4: Let $\{G_i\}_{i\in I}$ be a family of supra pre-connected subsets of X. If there is j ∈ I such that G_i and G_j are not supra pre-separated for each i ≠ j, then $\cup_{i\in I}G_i$ is supra pre-connected.

Proof: Let (X, μ) be a supra topological space. Suppose that $\cup_{i\in I}G_i$ is not supra pre-connected. Then there exist non-empty supra pre-separated sets A and B of X such that $\cup_{i\in I}G_i$ = A ∪ B. For each i ∈ I, G_i is supra pre-connected and G_i ⊆ A ∪ B. Then by Lemma 2.13, either G_i ⊆ A and G_i ∩ B = ∅ or else G_i

$\subseteq$ B and $G_i \cap A = \emptyset$. If possible, let for some r,s $\in$ I with r $\neq$ s, $G_r \subseteq A$ and $G_s \subseteq B$. Then $G_s \cap Cl_p^{\mu}(G_r) \subseteq A \cap Cl_p^{\mu}(B) = \emptyset$. Similarly, $Cl_p^{\mu}(G_s) \cap G_r = \emptyset$. Hence G_r, G_s are supra pre-separated, which is a contradiction to our hypothesis. Thus either $G_i \subseteq A$ and $G_i \cap B = \emptyset$, for each i $\in$ I or else $G_i \subseteq B$ and $G_i \cap A = \emptyset$, for each i $\in$ I. In the first case, we get $\cup_{i\in I} G_i \subseteq A$ and $B = \emptyset$ and in the second case, we get $\cup_{i\in I} G_i \subseteq B$ and $A = \emptyset$. Therefore, either $A = \emptyset$ or $B = \emptyset$, none of which is true. Thus $\cup_{i\in I} G_i$ is supra pre-connected.

Theorem 3.5: A non-empty subset G of X is supra pre-connected if and only if for any two elements x and y in G there exists a supra pre-connected set H such that x, y $\in$ H $\subseteq$ G.

Proof: Suppose G is supra pre-connected. Let x, y $\in$ G. Since G is itself a supra pre-connected set, we have G is a supra pre-connected set such that x, y $\in$ G $\subseteq$ G. Conversely, assume that the given condition holds. Suppose by contrary that G is not supra pre-connected. Then there exist two non-empty supra pre-separated sets P and Q in X such that $G = P \cup Q$. Choose x $\in$ P and y $\in$ Q. So x, y $\in$ G and hence by hypothesis there exists a supra pre-connected set H such that x, y $\in$ H $\subseteq$ G. Since $H \cap P$ and $H \cap Q$ are non-empty subsets of the supra pre-separated sets P and Q, respectively, by Lemma 2.12, we have $H \cap P$ and $H \cap Q$ are also supra pre-separated sets of X. Thus $H\cap P$ and $H \cap Q$ are non-empty supra pre-separated sets with $H = (H \cap P) \cup (H \cap Q)$. This is contrary to the supra pre-connectedness of H. Thus G is supra pre-connected.

IV. CONCLUSION

In this paper we have studied some properties of supra pre-connected set in topological spaces.

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