Semi prime ideals in Ordered Meet Hyperlattices

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Abstract- *In this article we discuss about the semi prime ideals in ordered meet hyperlattices.*

I. INTRODUCTION

In this paper, we consider order relation ≤ as x ≤ y if and only if y = x V y for all x, y Є L, and we introduce semi prime ideals in ordered meet hyperlattices. Here, we give some results about them.

I. Preliminaries

Definition 1.1:

Let H be a non-empty set. A Hyperoperation on H is a map ∘ from H×H to P*(H), the family of non-empty subsets of H. The Couple (H, ∘) is called a hypergroupoid . For any two non-empty subsets A and B of H and x Є H, we define A ∘ B = $\cup_{a\in A, b\in B}$ a ∘ b;

A ∘ x = A ∘ {x} and x ∘ B = {x} ∘ B

A Hypergroupoid (H, ∘) is called a Semihypergroup if for all a, b, c of H we have (a ∘ b) ∘ c = a ∘ (b ∘ c). Moreover, if for any element a ∈ H equalities

A ∘ H = H ∘ a = H holds, then (H, ∘) is called a Hypergroup.

Definition 1.2:

Let L be a non-empty set, **∧:** L × L → p* (L) be a hyperoperation and V : L × L →L be an operation. Then (L, V, ∧) is a Meet Hyperlattice if for all x, y, z Є L. The following conditions are satisfied:

1) x Є x ∧ x and x = x V x
2) x V (y V z) = (x V y) V z and x ∧ (y ∧ z) = (x ∧ y) ∧ z
3) x V y = y V x and x ∧ y = y ∧ x
4) x Є x ∧ (x V y) ∩ x V (x ∧ y)

Definition 1.3:

An Ideal [1] P of a meet hyperlattice L is Prime [2] if for all x, y Є L and x V y Є P, we have x Є P and y Є P.

Proposition 1.4:

Let L be a meet hyperlattice. A subset P of a hyperlattice L is prime if an only if L\P is a subhyperlattice of L.

Definition 1.5:

Let (L, V, ∧, ≤) be an ordered meet hyperlattice and I ⊆ L be an ideal and F be a filter of L. We call I is a semiprime ideal if for every x, y, z Є L, (x V y) Є I or (x V z) Є I implies that x V (y ∧ z) ⊆ I. Also, we call F is a semiprime filter if x ∧ y ⊆ F or x ∧ z ⊆ F implies that x ∧ (y V z) ⊆ F.

II. Properties of semi prime ideals in ordered meet hyperlattices [4]

Every Prime ideal I is semi prime [3]. Since if (x V y) Є I or (x V z) Є I, we have x Є I and y Є I or x Є I and z Є I.
If x ЄI, by x V (y ∧ z) ≤ x we have, x V (y ∧ z) ⊆ I
Otherwise, we have y, z Є I.
So, y ∧ z ⊆ I and x V (y ∧ z) ⊆ I.

Proposition 2.1:

Let (L, V, ∧, ≤) be an ordered meet hyperlattce and I be a semiprime ideal of L. Also, for any A, B ⊆ L, A ≤ B ⊆ I implies that A ⊆ I. Then, I_1 = {J Є Id(L); J ⊆ I} is a semiprime ideal of L. If L is a finite hyperlattice, I_2 = ∪ {J; J ⊆ I} is a semiprime ideal of L.

Proof:
Let J_1, J_2 ⊆ I, then $J_1 \wedge J_2$ ⊆ I ∧ I.
Since, I is an ideal of L, we have I ∧ I ⊆ I.
Therefore, $J_1 \wedge J_2$ ⊆ I.
Let J_1 V J_2 ⊆ I_1, J_1 V J_3 ⊆ I_1 for any J_1, J_2, J_3 Є Id (L).
Then, let x' ∈ J_1 V ($J_2 \wedge J_3$).
x' = x V y for x Є J_1, y Є $J_2 \wedge J_3$.
Therefore, y = y' ∧ y'' for some y' Є J_2 or y'' Є J_3.
We have x V y' Є $J_1 V J_2$ ⊆ I or x V y'' Є $J_1 V J_3$ ⊆ I.
Since I is semiprime, we have
x V (y' ∧ y'') ⊆ I and J_1 V ($J_2 \wedge J_3$) ⊆ I.
If L is finite, we prove that I_2 is a semi prime ideal.
Let x, y Є I_2.
Thus, x Є J_1 ⊆ I or y Є J_2 ⊆ I.
Therefore, x ∧ y ⊆ $J_1 \wedge J_2$ ⊆ I.
Let x ≤ y Є J_1 ⊆ I.
Since, I is an ideal, we have
x Є I or x Є I_2.
Since L I finite, I_2 is a semiprime ideal of L.

Theorem 2.2:
Let L be a s-good (x ∧ 0 = x) bounded ordered meet hyperlattice and I be an ideal and F be a filter of L such that

I∩F = ∅ and for any A ⊆ F. If F is a semiprime filter, there exists a semiprime ideal J such that I ⊆ J and J∩F = ∅.

Proof:

Let F be a semiprime filter and θ be a congruence on L which is defined as a θ b if and only if
F:a = F:b where F:a = {x ∈ L; a ∧ x ⊆ F}.
Then, θ is an equivalence relation.
Now, we show that θ is compatible with ∧ and V.
Let a θ b, since F is a semiprime filter, we have
F:a V c = (F:a) ∪ (F:c)
= (F:b) ∪ (F:c)
= F:b V c.
Thus, a V c θ b V c.
Let y ∈ F:a ∧ c.
Thus, y ∧ a ∧ c ⊆ F and therefore,
y ∧ c ⊆ F:a = F:b.
y ∧ c ∧ b ⊆ F and y ∈ F:c ∧ b.
Therefore, θ is compatible with ∧.
Clearly, θ is a strongly regular relation and therefore L/θ is a lattice.
Now, we claim that L/θ is a distributive lattice.
Let s θ x ∨ (y ∧ z) and
u ∈ F:s = F:x ∨ (y ∧ z).
A = u ∧ (x ∨ (y ∧ z)) ⊆ F.
Since L is bounded, we have A ≤ u ∧ (1 ∨ (y ∧ 1)) ≤ u ∧ (y ∧ 1).
So, we have u ∧ y ⊆ F or
u ∧ x ⊆ F.
By semi prime property of F, we have
u ∧ (x ∨ y) ⊆ F and since
u ∧ (x ∨ y) ≤ u ∧ (x ∨ y) ∧ (x ∨ z).
Therefore, u ∈ F:(x ∨ y) ∧ (x ∨ z) and
L/θ is a distributive lattice.
Also, in L/θ, we have Iθ ∩ Fθ = φ.
If there exists y ∈ Hθ ∩ Fθ, we have I θ F.
Thus, F:I = F:F and since 0 ∧ F = 0 ⊆ F, we have 0 ∈ F:I. 0 ∧ I = 0 ⊆ F which is a contradiction to I ∩ F = φ.
So Iθ ∩ Fθ = φ. Since I ∩ F = φ, there exists Pθ ∈ L/θ such that Iθ ⊆ Pθ where Pθ is a prime ideal.
Let us consider a canonical map h: L ⟶ L/θ by h(a) = θ(a).
Therefore, we have I ⊆ h −1 (Pθ) = P,
P ∩ F = φ and
P is a prime ideal of L.

Theorem 2.3:

Let (L, ∨, ∧ , ≤) be an ordered meet hyperlattice. L is a distributive hyperlattice if and only if for every ideal I and filter F of L such that I ∩ F = φ, there exist ideal J and filter G of L such that I ⊆ J, F ⊆ G, J ∩ G = φ, J or G is semi prime and for every x ∈ L, we have x ∈ J ∪ G

Proof:

Let L be a distributive hyperlattice. We know that, if (L, ∨, ∧) is a distributive hyperlattice if I and F are ideal and filter, respectively then I ∩ F = φ, then there exist ideal J and filter G of L such that
I ⊆ J, F ⊆ G, then J ∩ G = φ.
Now, we show that L is distributive.
Let x, y, z ∈ L and I be the ideal which is generated by
(x ∨ y) ∧ (x ∨ z) and F be a filter which is generated by x ∨ (y ∧ z).
Let, x ∨ (y ∧ z) ≰ (x ∨ y) ∧ (x ∨ z).
Therefore I ∩ F = φ.
Then, there exist ideal J and filter G such that I ⊆ J and F ⊆ G, J ∩ G = φ.
If J is semi prime ideal, since
x ∨ y ∈ J or x ∨ z ∈ J, we have x ∨ (y ∧ z) ⊆ J.
Since x ∨ (y ∧ z) ⊆ G, we have J ∩ G ≠ φ which is a contradiction.
If G is semi prime, we have x ∈ G or y ∧ z ⊆ G.
If y ∈ G, since x ∈ G, we have x ∨ y ∈ G, and if z ∈ G, we have x ∨ z ∈ G,
which is a contradiction to J ∩ G = φ.
So neither y nor z are not in G.
If both y, z ∈ J, y ∧ z ⊆ J.
This is contradiction with J ∩ G = φ.
So both y, z ∈ J is impossible.
Let y not belongs to J and z ∈ J.
We have x ∨ z ∈ J.
Since, x ∨ y ≤ (x ∨ y) ∧ (x ∨ z) ∈ J,
we have x ∨ y ∈ J.
But x ∨ y ∈ G, and this is contradiction.
Then, we have x ∨ (y ∧ z) ≤ (x ∨ y) ∧ (x ∨ z).
Let (x ∨ y) ∧ (x ∨ z) ≰ x ∨ (y ∧ z) and I is an ideal which is generated by x ∨ (y ∧ z), F is a filter which is generated by (x ∨ y) ∧ (x ∨ z).
Similarly, we arrive at the contradiction and the proof is completed.

II. CONCLUSION

In this paper we have discussed about the semi prime ideals and their properties in ordered meet hyperlattices.

REFERENCES

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[2] https://en.wikipedia.org/wiki/Prime_ideal

[3] http://mathworld.wolfram.com/SemiprimeIdeal.html
[4] https://www.researchgate.net/publication/317743259_Ordered_join_hyperlattices