A Study of Knot Theory

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Abstract- *In the past 50 years, knot theory, a branch of three dimensional geometry has become an extremely well-developed subject. In knot theory, the most important connection results are derived from a construction that assigns to the fundamental group of the knot. In this article combinatorial methods are applied to develop knot groups.. This paper concentrates on knots and a detailed study of the equivalence of knots using some examples. Some theorems and prepositions are discussed using the knottedness.This paper is focussed to understand knot theory through a discussion of research problems.Each section deals with a specific problem, or with an area in which problems exist. The results reflect few problems in graph theory (such as the tricolour , four color problem) that are really problems in the theory of knots. As a result we analyze the possible symmetry groups of two component links and its properties.*

Keywords- Symmetric Groups, Knot Groups, Linking Number

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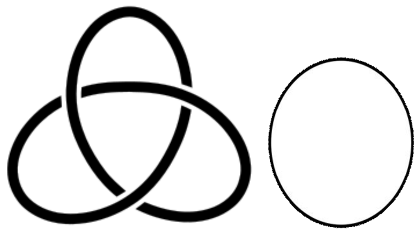
I. INTRODUCTION

Man has been fascinated by Knots since time immemorial. He has been using different types of knots for different purposes. He used knots to make nets, tie things together, weave clothes and mat; construct bridges, climb rocks and so on. Beautiful decorative items and embroidery works are being made using knots. The Inca dynasty of South America used Quipu (word meaning knot) to store information to be sent to different places. Man's interest in knots gave it a symbolic significance in certain cultures and spirituality. A link called the Borromean Rings was a symbol of the Borromeo family, an aristocratic family of Northern Italy. The endless knot or eternal knot is a symbolic knot important as a cultural marker in Tibet. In India, a ring made of Dharbha Grass is worn at the time of certain rituals. The ring is made by a special knot called the Pavithrakettu. A scout boy plays with a rope tying many kinds of knots like the overhand knot, the Reef knot, the figure eight knot, the clove hitch, the fisherman knot, the square knot and else. When Alexander the Great cut the highly complicated Gordian knot with his sword to untie the knot, it found its place in English literature as a metaphor for an intractable problem solved easily by loopholes.

II. PRELIMINARIES

2.1 Knot: A subset K of R^3 is a knot if there exists a homeomorphism of unit circle C into three dimensional space R^3 whose image is K.



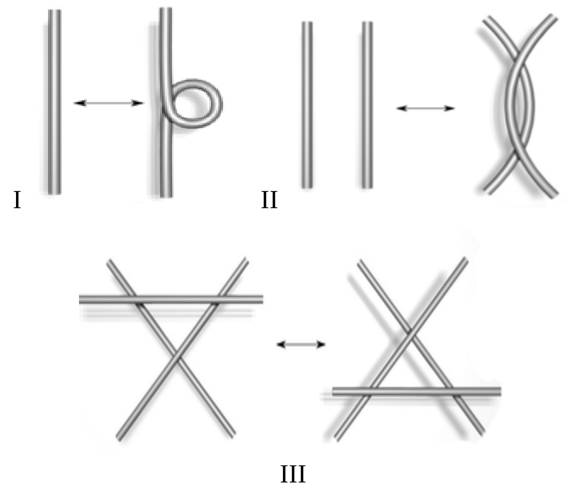
The unknot *Trefoil knot*

2.2. Equivalence : Two knots K_1 and K_2 are said to be equivalent if there exists a homeomorphism of R^3 onto itself which maps K_1 onto K_2 . If two knots are equivalent, then they are said to be of the same type. Any knot equivalent to an unknot is of trivial type

2.3 .Knot invariant: Let K be a knot and let a specific quantity ρ (K) be assigned to it. Then ρ (K) is said to be a knot invariant if it is equal for any knot equivalent to *K.*

2.4. Reidemeister move : A Reidemeister move is one of the three ways to change a projection of the knot that will change the relation between the crossings.

- Twist and untwist in either direction.
- Move one strand completely over another.
- Move a strand completely over or under a crossing.



2.5. Crossing number: The minimum number of crossings points of all the regular diagrams of a knot K, denoted by c(K) is called the crossing number of K. A regular diagram of K that has c(K) crossing points is the minimum regular diagram of K.

2.6. Link: A link is a finite ordered collection of knots that do not intersect each other. Each knot is said to be a component of the link.

2.7. Linking number: Suppose that D is an oriented regular diagram of a two component link L= { K_1,K_2}. Also, suppose that the crossing points of D at which the projections of K_1 and K_2 intersect (self - intersections of each component not considered) be C_1 ,C_2,C_3........C_m . Then (signC_1+signC_2+.....+signC_m } is called the linking number of K_1 and K_2 and is denoted by lk (K_1,K_2).

2.8. Colouring: A strand in a projection of a link is a piece of the link that goes from one undercrossing to another with only overcrossings in between. A projection of a knot or link is tricolourable if each of the strands in the projection can be coloured one of three different colours so that at each crossing either 3 different colours come together or all the same colours come together. At least two of the colours must be used.

III. METHODOLOGY

THEOREM 3.1

c(K) = $\min_{D\epsilon \mathcal{D}}$ c(K) is a knot invariant, where D is the set of all regular diagrams D of K.

PROOF:

Suppose that D_0 is the minimum regular diagram of a knot K. Let K′ be a knot equivalent to K and let D_0’ be a minimum regular diagram of K′. Since D_0’ is also a regular diagram of K, by [2.5]

$$c(K) \leq c(D_0) \leq c(D_0')$$

Similarly, D_0 is a regular diagram of K′ so that

$$c(K') \leq c(D_0') \leq c(D_0)$$

Thus, $c(D_0)=c(D_0')$. Hence $c(D_0)$ is the minimum number of crossing points of the knots equivalent to K. Consequently, c(K) is invariant.

THEOREM 3.2

The linking number lk (K_1,K_2) is an invariant of the link L{ K_1,K_2}.

PROOF:

Any projection of a link can be obtained from another by a sequence of Reidemester moves. So it is enough to prove that the Reidemester moves do not change the linking number.

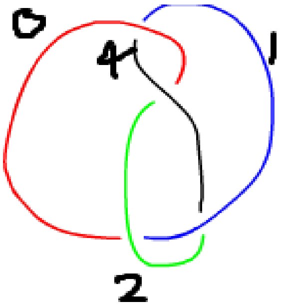
The first Reidemester move can create or eliminate a self – crossing in one of the components. It will not affect the crossing involving both the components. Thus the first Reidemester move leaves the linking number unaltered.

To show the effect of Type II Reidemester move, consider two strands from different components with certain orientation. One of the new crossings contribute a +1 to the sum while the other a -1. Thus the net contribution to the linking number is 0. Even if the orientation of one strand is changed, the net effect is 0.

THEOREM 3.3

The figure-eight knot is 5 – colourable but not 3 colourable.

PROOF:



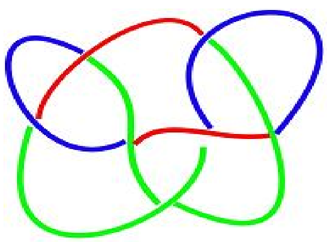
The possible choices of λ_r ,λ_s and λ_k are (1, 4, 0), (4,1,0), (2,3,0), (3,2,0), (0,2,1), (1,1,1), (2,0,1), (0,4,2), (2,2,2), (4,0,2), (0,1,3), (1,0,3), (0,3,4), (1,2,4), (2,1,4), (3,0,4) where $\lambda_i \in \{0,1,2,3,4\}$. Thus choosing values from this set it can be shown that figure-eight knot is 5-colourable.

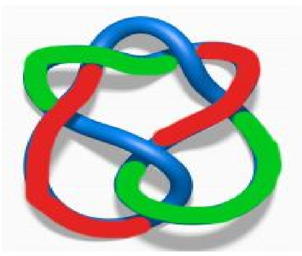
If $\lambda_i \in \{0,1,2,3,4\}$. the possible values of λ_r ,λ_s and λ_k can be taken so that the required condition is satisfied are (1,2,0), (2,1,0), (1,1,1,), (0,2,1), (2,0,1), (0,1,2), (1,0,2) and (2,2,2).

Suppose the overstrand through the crossing point at P_1 is assigned] = =] Z = 0. Then at this point $\lambda_r = 1$,$\lambda_s = 2$ or vice versa. Let the overstrand at the crossing point P_2 be assigned 1 and that at P_4 be 2. At P_2 , ,λ_r and ,λ_s can take values 0 and 2 only. But this assignment is not possible since the overstrand at P4 is assigned 2. By a similar argument other possibilities are also ruled out. Hence figure 8 knot is not tricolourable. But it is 5 – colourable.

IV. EXAMPLES

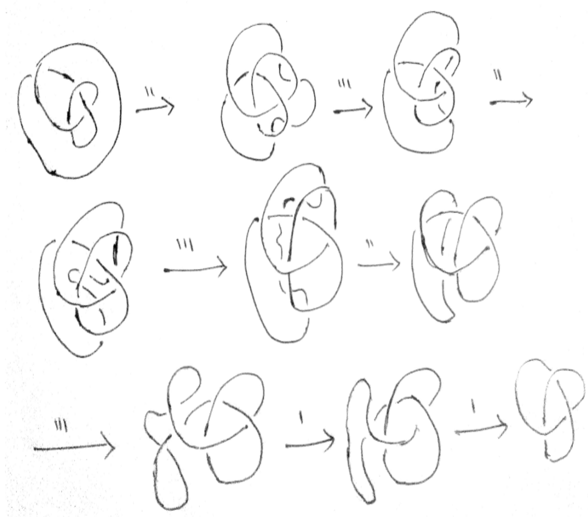
1.1. The six crossings knot projection is tricolourable. The projection of a seven crossings tricolourable knot is shown .





1.2. The linking number of the link in figure is $\frac{1}{2}\{+1 + 1 + 1 + 1\}=2$.

1.3. The figure- eight knot is equivalent to its mirror image.
Solution:



The problem is solved using the Reidemeister moves

1. The figure eight knot is changed by moving one strand completely over the other (II)
2. Now using the Reidemeister move (III) a strand is moved under the other
3. Continuously using the three moves finally we arrive the mirror image of figure eight knot.

Hence figure eight knot is equal to its mirror image.

V. CONCLUSION

Thus this paper helped to study the basic of knots. Keeping this paper as basic we would futhur study about the knot equivalence , knot invariance ,knot colouring and knot tabulation in detail.

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