Observation on $y^2 = 11x^2 + 1$

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Abstract- *The binary quadratic equation* $y^2 = 11x^2 + 1$ *is analyzed for its distinct integer solutions and we obtain infinitely many Pythagorean triangles. A few interesting relations among the sides are also given.*

Keywords- The binary quadratic equation, Pell equation, Pythagorean triangles, Integer Solutions.

I. INTRODUCTION

Number theory is a branch of pure mathematics dedicated mainly to the study of the integers. Mathematics is the queen of the sciences and number theory is the queen of mathematics. The Diophantine equation is of the form $x^2 = dy^2 + 1$ where d a nonsquare positive integer is and integer solutions sought for x and y. This equation was first studied in India, starting with Brahmagupta, who developed the Chakravala method to solve Pell's equation. Pell's equation has infinitely many distinct integer solutions when $'d'$ takes different numerical values.

In this communication, by applying the non-trivial integral solutions of the binary quadratic equation $y^2 = 11x^2 + 1$, we obtain infinitely many Pythagorean triangles. The recurrence relations satisfied by the sides of the triangle are presented.

II. METHOD OF ANALYSIS

Consider the binary quadratic equation

$$y^2 = 11x^2 + 1$$

whose general solution (x_n, y_n) is represented by

$$x_n = \frac{1}{2\sqrt{11}}\left[\left(10+3\sqrt{11}\right)^{n+1} - \left(10-3\sqrt{11}\right)^{n+1}\right]$$

$$y_n = \frac{1}{2}\left[\left(10+3\sqrt{11}\right)^{n+1} + \left(10-3\sqrt{11}\right)^{n+1}\right], \quad n = 0,1,2,3,\ldots$$

Assuming $x_n = b_n$ and $y_n = a_n - \frac{10}{3}b_n$, we obtain

$$a_n = \frac{1}{6\sqrt{11}}\left[\left(10+3\sqrt{11}\right)^{n+2} - \left(10-3\sqrt{11}\right)^{n+2}\right]$$

$$b_n = \frac{1}{2\sqrt{11}}\left[\left(10+3\sqrt{11}\right)^{n+1} - \left(10-3\sqrt{11}\right)^{n+1}\right],$$

where $n = 0,1,2,3,\ldots$

Considering a_n, b_n as the generators of a Pythagorean triangle, its legs X_n, Y_n and hypotenuse Z_n are found to be

$$X_n = 2a_nb_n = \frac{1}{66}\left[\left(10+3\sqrt{11}\right)^{2n}\left(3970+1197\sqrt{11}\right) + \left(10-3\sqrt{11}\right)^{2n}\left(3970-1197\sqrt{11}\right) - 20\right]$$

$$Y_n = \frac{1}{396}\left[\left(10+3\sqrt{11}\right)^{2n}\left(77410+23340\sqrt{11}\right) + \left(10-3\sqrt{11}\right)^{2n}\left(77410-23340\sqrt{11}\right) + 16\right]$$

$$Z_n = \frac{1}{396}\left[\left(10+3\sqrt{11}\right)^{2n}\left(80992+24420\sqrt{11}\right) + \left(10-3\sqrt{11}\right)^{2n}\left(80992-24420\sqrt{11}\right) - 20\right]$$

The above values of X_n, Y_n and Z_n satisfy the following recurrence relations:

$$X_{n+2} - 398X_{n+1} + X_n = 80$$

$$Y_{n+2} - 398Y_{n+1} + Y_n = -16$$

$$Z_{n+2} - 398Z_{n+1} + Z_n = 20$$

The following table shows a few numerical Pythagorean triangles:

Table 1. Numerical Examples

n	X_n	Y_n	Z_n
1	47880	155601	162801
2	19056240	61928791	64794409
3	7584335760	24647503200	25788012000

III. CONCLUSION

In this paper, we have presented infinitely many Pythagorean triangles for the considered binary quadratic equation $y^2 = 11x^2 + 1$. As the binary quadratic equations are rich in variety, one may search for the other solutions of the considered binary quadratic equations and determine their integer solutions along with suitable properties.

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