Rational Cubic Trigonometric B´Ezier Curve With Tension Parameter

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Abstract- *In this paper, Rational cubic trigonometric Bezier´ curve with three shape parameters in which one is tension parameter and two are shape parameters, which is analogous to tension quadratic trigonometric Bezier´ curve with two shape parameter, is presented. The presence of tension and shape parameters provides a local control on the shape of the curve which enables the designer to control the curve more than the ordinary Bezier´ curve. Here we include one tension parameter which can be useful to better visualize objects and graphics. Some applications are given by examples.*

Keywords- Trigonometric Bezier´ basis functions, Cubic trigonometric Bezier´ curve, Shape parameters, Tension parameter, shape control of the curves.

I. INTRODUCTION

Curves and surfaces design is an important research topic of CAGD (Computer Aided Geometric Design) and CG (Computer Graphics). To obtain satisfactory graphics, it is required to improve the shape of the curve so that it can approximate the control polygon. Spline functions are universally recognized as highly effective tool in a variety of applications in many areas of science and technology, especially CAGD and CG. Since trigonometric functions can be used to create spline curves that are, in many ways, superior to the more common splines. If one wishes to modify the shape of the curve, the control points need to be changed. To modify the shape of the curve under the condition that the control polygon is not changed, shape parameters are required. Therefore in my paper, I propose to study on shape parameters and tension parameter, which are affect the curve segment as little as possible and has a predictable adjusting role on the curves and surfaces. Lamnii et. al. discussed a tension quartic trigonometric Bezier´ curve with tension parameter. Now keeping in this, we can extend the idea of Lamnii, and analogous to tension quadratic trigonometric Bezier´ curve with two shape parameters, a newly constructed rational cubic trigonometric Bezier´ curve with three shape parameters, where we include a tension parameter is presented in this paper. The proposed curve inherits all trigonometric properties of the traditional Bezier´ curve and is used to construct open and closed curves.

The remainder of this paper is organized as follows. In section 2, the cubic trigonometric Bezier´ basis functions with shape parameters and tension parameter are defined and the properties of these functions are shown. In section 3, rational cubic trigonometric Bezier curves are given and their properties are discussed. Shape control of the curve by using shape parameters and tension parameter, is discussed in section 4. In section 5, the representation of ellipse and circle are given. The approximability of the rational cubic trigonometric Bezier´ curves with three shape parameters and cubic Bezier´ curves corresponding to their control polygon are discussed in Section 6. Conclusion is given in section 7.

II. CUBIC TRIGONOMETRIC BEZIER BASIS FUNCTIONS WITH THREE SHAPE PARAMETERS

Firstly, we can define the cubic trigonometric basis functions with tension and shape parameters and then their properties are given.

2.1 The construction of the cubic trigonometric basis functions with Tension and shape Parameters

The cubic trigonometric basis functions are given as follows:

Definition 2.1 for $t \in \left[0, \frac{\pi}{2\beta}\right]$, two arbitrarily real value of m, n and β be the tension parameter the following four functions are defined as cubic trigonometric Bezier basis functions $B_{i,\beta}$; for i = 0, 1, 2, 3, with tension parameter β and two shape parameters m, n:

$$B_{0,\beta}(t,m,n) = [1-\sin(\beta t)]^2[1-m\sin(\beta t)];$$
$$B_{1,\beta}(t,m,n) = \sin(\beta t)[1-\sin(\beta t)][2+m(1-\sin(\beta t))];$$
$$B_{2,\beta}(t,m,n) = \cos(\beta t)[1-\cos(\beta t)][2+n(1-\cos(\beta t))];$$
$$B_{3,\beta}(t,m,n) = [1-\cos(\beta t)]^2[1-m\cos(\beta t)];$$

(1)

for m = n = 0, the basis functions are quadratic trigonometric polynomials.

For m, n $\neq$ 0, the basis functions are cubic trigonometric polynomials. Figure 1, plots these basis functions for different values of the tension parameter β in the interval $t \in [0, \pi/2\beta]$.

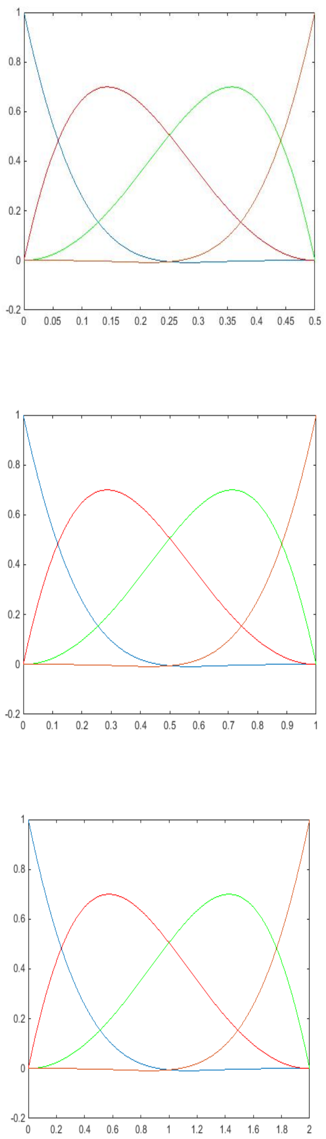


Figure 1. The curves of blending basis functions for $\beta = \pi$, $\beta = \frac{\pi}{2}$, and $\beta = \frac{\pi}{4}$

2.2 The properties of cubic trigonometric basis functions

Theorem 1. The basis functions $B_{i,\beta}(t,m,n),(i=0,1,2,3)$ defined in eq.(1) have the following properties:

(a) Non-negativity: When $t \in \left[0, \frac{\pi}{2\beta}\right]$ there are $B_{i,\beta}(t,m,n) \geq 0,(i=0,1,2,3)$.

(b) Partition of Unity**:** One has $\sum_{i=0}^{3} B_{i,\beta}(t,m,n) = 1;$

(c) Symmetry**:**

$$B_{i,\beta}(t,m,n) = B_{3-i,\beta}\left(\frac{\pi}{2\beta} - t,m,n\right); \text{ for } i = 0,1,2,3;$$

(d) Maximum: Each $B_{i,\beta}(t,m,n)$ has one maximum value in $t \in \left[0, \frac{\pi}{2\beta}\right]$

(e) Monotonicity**:** For a given value of shape parameter m and n, $B_{0,\beta}$ is monotonically decreasing and $B_{3,\beta}$ is monotonically increasing.

III. RATIONAL CUBIC TRIGONOMETRIC BEZIER CURVES WITH TENSION PARAMETER

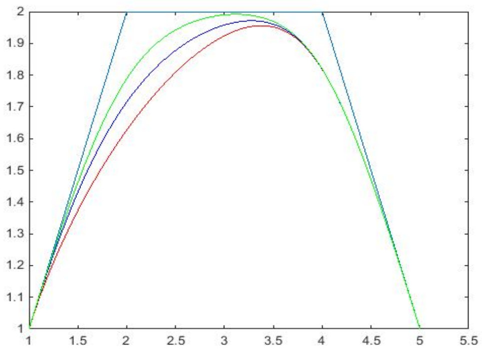
3.1 The construction of the rational cubic trigonometric Bezier curves with tension Parameter

Definition 3.1 Let P_i (i = 0, 1, 2, 3) in R^2 or R^3 , Then rational cubic trigonometric Bezier curves with tension parameter $\beta > 0$ and $t \in \left[0, \frac{\pi}{2\beta}\right]$ is defined as:

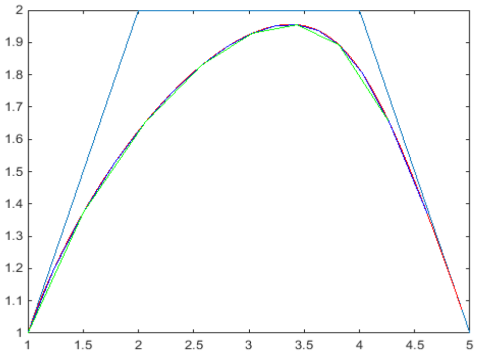
$$C_\beta\ (t,m,n) = \frac{\sum_{i=0}^{3} w_i\ B_{i,\beta}(t,m,n)\ P_i\ ;}{\sum_{i=0}^{3} w_i B_{i,\beta}\ (t,m,n)\ ;} \qquad \textbf{(2)}$$

Where $m,n \in [0,2]$ The points P_i (i=0, 1, 2, 3) in R^2 or R^3 are called rational cubic trigonometric Bezier control points and $B_{i,\beta}(t,m,n)$ are the cubic trigonometric Bezier basis functions defined in eq.(1).

Figure 2. we can show that the rational cubic Trigonometric Bezier curves with different tension parameter values. It observed that the rational cubic trigonometric Bezier curves with shape and tension parameter are close to the control polygon. Thus, the rational cubic trigonometric Bezier curves with shape and tension parameters can nicely preserve the feature of the control polygon. The effect of this tension factor β is illustrated in Figures 3.



(a) m=-1,0, 1; n=1.5; $\beta = \pi/4$



(b) different value of β =2, 10 and 20 ;

Figure 2. The cubic Trigonometric Bezier curves (for $t \in [0, \pi/2\beta]$)

3.2 The properties of the rational cubic trigonometric Bezier curves with Tension Parameter

Theorem 2. The rational cubic trigonometric Bezier curves with shape and tension parameters (2) have the following properties:

(i) End point properties:

$$C_\beta(0, m, n) = P_0;$$

$$C_\beta\left(\frac{\pi}{2\beta}, m, n\right) = P_3;$$

$$C'_\beta(0, m, n) = \frac{w_1}{w_0}(2+m)\beta(P_1 - P_0);$$

$$C'_\beta\left(\frac{\pi}{2\beta}, m, n\right) = \frac{w_2}{w_3}(2+n)\beta(P_3 - P_2);$$

(3)

(ii) Symmetry: The control points P_i and P_{3-i} (for i =0, 1, 2, 3) define the same rational cubic trigonometric B´ezier curve in different parameterizations, i.e. for i = 0, 1, 2, 3; $t \in \left[0, \frac{\pi}{2\beta}\right]$ and $m, n \in [0, 2]$;

$$C_\beta(t, m, n, P_i) = C_\beta\left(\frac{\pi}{2\beta} - t, n, m, P_{3-i}\right)$$

$$C_\beta(t; P_0; P_1; P_2; P_3) = C_\beta\left(\frac{\pi}{2\beta} - t; P_0; P_1; P_2; P_3\right)$$

(4)

(iii) Geometric invariance: The shape of the curve (4.2) is independent of the choice of its control points i.e. the curve (2) satisfies the following two equations for i = 0, 1, 2, 3.

$$C_\beta(t, m, n, P_i + q) = C_\beta(t, m, n, P_i) + q;$$

$$C_\beta(t, m, n, P_i \times T) = C_\beta(t, m, n, P_i) \times T;$$

(5)

where $t \in \left[0, \frac{\pi}{2\beta}\right]$, $m, n \in [0, 2]$, q is any arbitrary vector in R^2 and T is an arbitrary 2 × 2 matrix.

(ii) **Convex hull**: The entire curve (2) is contained within the convex hull of its defining control points P_i for i = 0, 1, 2, 3.

(iii) **Variation diminishing property**: No straight line intersects a Bezier curve more times than it intersects its control polygon.

IV. SHAPE CONTROL OF THE RATIONAL CUBIC TRIGONOMETRIC BEZIER CURVE

For control points $P = (P_0, P_1, P_2, P_3)$ be a set of points P_i (i = 0, 1, 2, 3)$\in$ R^2 or R^3, $t \in \left[0, \frac{\pi}{2\beta}\right]$ and $m, n \in [0, 2]$, From the eq. **(2)** the rational cubic trigonometric Bezier curves with tension parameters β > 0 associated with the set P is defined as:

$$C_\beta(t,m,n) = \frac{\sum_{i=0}^{3} w_i B_{i,\beta}(t,m,n) P_i;}{\sum_{i=0}^{3} w_i B_{i,\beta}(t,m,n);}$$

The parameters m and n controls the shape of the curve (2). In figure 3, the cubic trigonometric Bézier curve with tension parameter gets closer to the control polygon as the values of the parameters m and n increases and curve has becomes smoother if tension parameter β increases.

In figure 3, the curves are generated by setting the values of $m = -1.5$, $n = 1.5$ and $\beta = \pi$ (red lines), $m = 1$, $n = 1$ and $\beta = \pi/2$ (blue lines), $m = 1.5$, $n = -1.5$ and $\beta = \pi/4$ (green lines).

Figure 4 shows a relationship among, rational cubic trigonometric Bézier curve with tension parameter, the cubic trigonometric Bézier curve with shape parameter and the rational quadratic trigonometric Bézier curve.

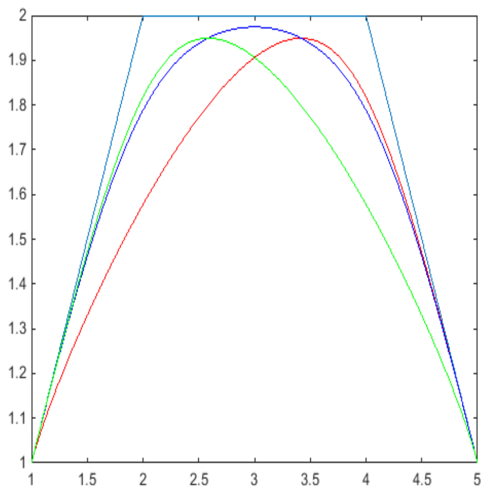


Figure 3. The effect on the shape of rational quadratic trigonometric Bézier Curves with tension parameter with altering the values of m, n and β.

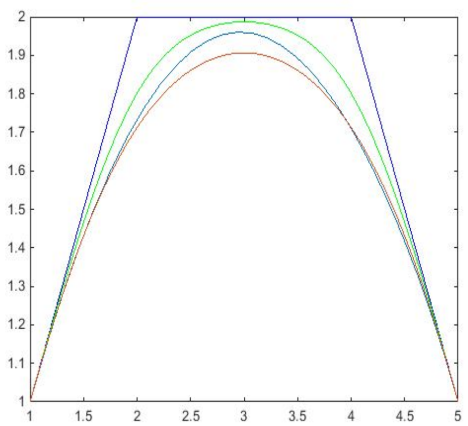
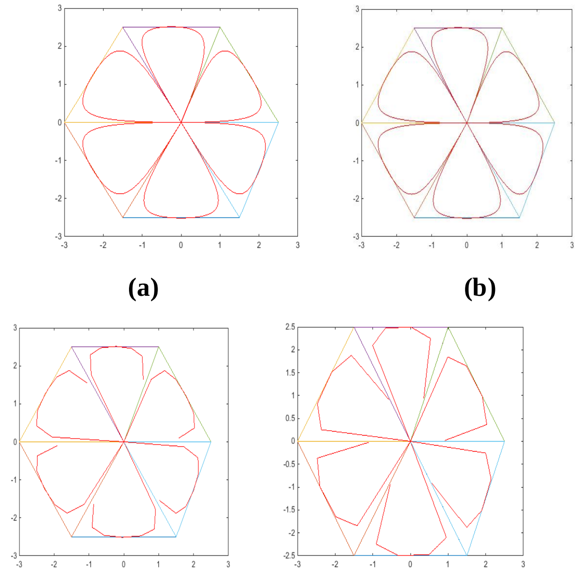


Figure 4. Relationship among, rational cubic trigonometric Bézier curve with tension parameter, the cubic trigonometric Bézier curve with shape parameter and the rational quadratic trigonometric Bézier curve.

In order to construct a closed cubic trigonometric Bézier curves with tension parameters. We can set $P_n = P_0$. In figure 4 (a), (b), (c), (d), we can construct the closed rational cubic trigonometric Bézier curves with tension parameter altering the values of the shape parameters m and n at the same time. The cubic trigonometric Bézier curves with tension parameter are generated by setting $m = 1.5$, $n = 1.5$ and $\beta = \pi$ in figure 4(a), $m = 1.5$, $n = 1.5$ and $\beta = \pi/4$ in figure 4(b), $m = 1.5$, $n = 1.5$ $\beta = 10$ in figure 4(c), $m = 1.5$, $n = 1.5$, $\beta = 20$ in figure 4(d).



(a) (b) (c) (d)

Figure 5. Rational Quadratic trigonometric Bézier Curves with tension parameter with altering the values of m, n and β.

V. THE REPRESENTATION OF AN ELLIPSE

Theorem 3. Let $P = (P_0, P_1, P_2, P_3)$ be four control points on an ellipse with semi axes $(2\sqrt{2})$ a and $(4\sqrt{2})$ b, by the proper selection of coordinates, their coordinates can be written in the form $P_0 = (2a, 0); P_1 = (a, 2b); P_2 = (-a, 2b); P_3 = (-2a, 0);$ Then the corresponding rational cubic trigonometric Bézier curve with tension and the shape parameters m=n=2 with β =0.8 and local domain t∈[0,4] represents arc of an ellipse with

$$x(t) = 2a(\cos(\beta t) - \sin(\beta t))$$
$$y(t) = 4b(\cos(\beta t) + \sin(\beta t) - 1)$$

Proof: If we take m = n =2 and $P_0 = (2a, 0); P_1 = (a, 2b); P_2 = (-a, 2b); P_3 = (-2a, 0);$ into (2), then the coordinate of quadratic trigonometric Bezier curve with tension and shape parameter are

$$x(t) = 2a(\cos(\beta t) - \sin(\beta t))$$
$$y(t) = 4b(\cos(\beta t) + \sin(\beta t) - 1)$$

This gives the intrinsic equation

$$\left(\frac{x(t)}{(2\sqrt{2})a}\right)^2 + \left(\frac{y(t)+4b}{(4\sqrt{2})b}\right)^2 = 1$$

It is an equation of an ellipse. Figure 6, Show an ellipse.

Corollary 1: According to theorem (3), if a=4.56 and b=2.28, then the corresponding rational cubic trigonometric Bézier curves with tension parameter $\beta = 0.8$, shape parameter m=n=2 and local domain $t \in [0, 4]$ represents arc of a circle. Fig 6, shows the Circle.

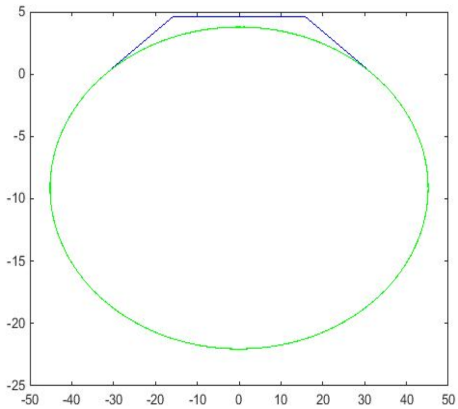


Figure 6. The representation of an ellipse

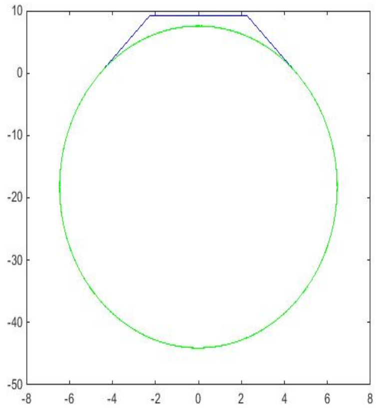


Figure 7. The representation of a circle.

VII. CONCLUSION

In this paper, we have presented the rational cubic trigonometric Bézier curve with tension parameter. All physical properties of rational cubic trigonometric Bézier curve with tension parameter are similar to the ordinary cubic Bézier curve. Due to the shape parameter, It is more useful in font designing as compared to ordinary cubic Bézier curve. We can deal precisely with circular arcs with the help of rational cubic trigonometric Bézier curve. The curve exactly represents the arc of an ellipse, the arc of a circle under certain conditions. Furthermore, it is analogous in structure to cubic Bézier curve. It is not difficult to adapt a rational cubic trigonometric Bézier curve with tension parameter to CAD/CAM system. The trigonometric polynomial blending functions constructed in this paper includes the tension parameter, which is mainly important for object visualization. The shape of the curve can be adjusted using the values of the shape and tension parameters. The applications indicate that, our method can be also applied to generate nice shape preserving spline curves. In future, it can be extended to construct basis functions by introducing more than one tension parameters.

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