# Analytical Solution of Unsymmetric Rectangular Laminated Composite Plate By FSDT Using Levy Approach

**Pundalik Bisse[1], Basavaraj Gudadappanavar[2] , Dr. R. J. Fernandes[3]**
[1] Dept of Civil Engineering
[2, 3] Assistant Professor, Dept of Civil Engineering
[1, 2, 3] SDMCET Dharwad

***Abstract***- *An exact solution to the deflection of unsymmetrical laminated plates is developed for a variety of boundary conditions. The procedure, based on a generalized Levy type solution considered in conjunction with the state space concept, is applicable to rectangular plates with two opposite edges simply supported and the remaining ones subjected to a combination of clamped, simply supported boundary conditions. The solutions are obtained for the first order shear deformation theory, Numerical results are presented for rectangular plates with different edge conditions, aspect ratios, and loadings.*

## I. INTRODUCTION

Laminated composite plates are becoming increasingly used in the aeronautical and aerospace industry as well as in other fields of modern technology. The accurate knowledge of deflection is an essential element in their design. In addition to the need for improved methods of analysis, there is an interest in the development of consistent shear deformation theories for these structures. A special case of unsymmetric laminates those which have an even number of orthotropic layers with principal material directions alternating at $0^0$ to $90^0$ to the laminate axes. Such laminates are called *unsymmetric cross ply laminates.*

In the this paper we present the levy type solution for deflection of unsymmetric cross ply laminates the levy type solution involves choosing a solution form that satisfies the simply supported boundary conditions on two parallel edges of a rectangular laminate, and then the partial differential equations of equilibrium are reduced to ordinary differential equations are then solved using the state-space approach, cross-ply plate strips under sinusoidal loading. One of the goals of this paper is the employment of a powerful analytical technique based on the state space concept allowing one to obtain exact Levy type solutions associated with the case of unsymmetric angle-ply and cross-ply laminated plates. This technique was used in a series of papers. Khdeir (1988,1989) developed an exact approach to the elastic state of stress and the free vibration of shear deformable unsymmetric angle-ply and cross-ply laminated plates, respectively. In the first-order shear deformation theory (FSDT), a constant state of transverse shear stresses is accounted for, an often the transverse normal stresses is neglected. The FSDT allow the computation of interlaminar shear stresses through constitutive which is quite simpler then deriving them though equilibrium equations. Comparisons with available exact solutions (obtained for simply supported edge conditions) are made, and appropriate conclusions concerning the various effects are formulated.

## II. GOVERNING EQUATIONS

The displacement model for unsymmetric laminate is given as follows:-

$$u(p,q,r) = u_0(p,q) + Z\theta\, p(p,q)$$

$$v(p,q,r) = v_0(p,q) + Z\theta\, q(p,q)$$

$$w(p,q,r) = w_0(p,q)$$

Where ( $u_0, v_0, w_0$) are the displacement components in the direction of p ,q ,r respectively of a point on the mid-plane (i.e., r=0)

Equations of equilibrium

$$\delta u_0; \quad \frac{\partial N_p}{\partial p} + \frac{\partial N_{pq}}{\partial q} = 0$$

$$\delta v_0; \quad \frac{\partial N_q}{\partial q} + \frac{\partial N_{pq}}{\partial p} = 0$$

$$\delta w_0; \quad \frac{\partial Q_p}{\partial p} + \frac{\partial Q_q}{\partial q} + P_r^+ = 0$$

$$\delta \theta_p; \quad \frac{\partial M_p}{\partial p} + \frac{\partial M_{pq}}{\partial q} - Q_p = 0$$

$$\delta \theta_q; \quad \frac{\partial M_q}{\partial q} + \frac{\partial M_{pq}}{\partial p} - Q_q = 0$$

For the static case, the governing equations appropriate for the displacement field and unsymmetric cross-ply laminate construction are given by

$$[C]\{\Delta\} = \{F\}$$

where the coefficients $C_{mn}$ for the first-order theory (FSDT), are listed below.

$[C] =$ stress resultants

$\{\Delta\} =$ deflections

$\{F\} =$ loadings

Where $\{\Delta\}^T = \{u_0, v_0, w_0, \phi_p, \phi_q\}$

$\{F\}^T = \{0, 0, q, 0, 0\}$

And the coefficients are defined as $C_{mn} = C_{nm}$

$C_{11} = A_{11}d_p^2 + A_{66}d_q^2;$ $C_{12} = (A_{12} + A_{66})d_p d_q;$ $C_{13} = 0;$

$C_{14} = B_{11}d_p^2 + B_{66}d_q^2;$ $C_{15} = (B_{12} + B_{66})d_p d_q;$ $C_{23} = 0;$

$C_{22} = A_{66}d_p^2 + A_{22}d_q^2;$ $C_{25} = B_{22}d_q^2 + B_{66}d_p^2;$ $C_{24} = C_{15};$

$C_{33} = -KA_{55}d_p^2 - KA_{44}d_q^2;$ $C_{34} = -KA_{55}d_p;$ $C_{35} = -KA_{44}d_q;$

$C_{44} = -KA_{55} + D_{11}d_p^2 + D_{66}d_q^2;$ $C_{45} = (D_{12} + D_{66})d_p d_q;$

$C_{55} = -KA_{44} + D_{22}d_q^2 + D_{66}d_p^2;$

## III. THE LEVY-TYPE SOLUTION

A generalized Levy-type solution, in conjunction with the state space concept is used to analyze the bending problem of unsymmetric cross-ply laminated rectangular plates. The Levy type solution can be developed for rectangular laminates with the following boundary conditions: The edges q= 0, b are simply supported, while the remaining ones (p = ±a/2) may have arbitrary combinations of clamped, and simply supported edge conditions (see Fig. 1). The generalized displacements may be expressed as products of undetermined functions and known trigonometric functions so as to identically satisfy the simply supported boundary conditions at q = 0, b:

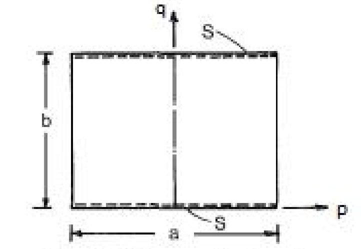


**Fig. 1 Plate coordinate system**

For this case one may be able to develop the Levy-type solutions for the unsymmetric cross-ply lamination scheme (see [18]). A uniformly distribution of the transverse load is considered, which for the present case takes the form

$$q(p,q) = \sum_{m=1}^{\infty} q_m \sin \frac{m\pi q}{b}$$

The displacement field is represented as

$$\begin{Bmatrix} u_0(p,q) \\ v_0(p,q) \\ w_0(p,q) \\ \phi_p(p,q) \\ \phi_q(p,q) \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m(p)\sin\beta q \\ V_m(p)\cos\beta q \\ W_m(p)\sin\beta q \\ P_m(p)\sin\beta q \\ Q_m(p)\cos\beta q \end{Bmatrix}$$

Where $\beta = \frac{m\pi}{b}$ ,m=1for all numerical problems

the displacement fields are substituting in stress resultants then we obtained second order partial differential equations

$$\ddot{U}_m = X_1 U_m + X_2 \dot{V}_m + X_3 \dot{W}_m + X_4 P_m + X_5 \dot{Q}_m$$

$$\ddot{V}_m = X_6 \dot{U}_m + X_7 V_m + X_8 W_m + X_9 \dot{P}_m + X_{10} Q_m$$

$$\ddot{W} = X_{11} W_m + X_{12} \dot{P}_m + X_{13} Q_m - q_m / KA_{55}$$

$$\ddot{P}_m = X_{14} U_m + X_{15} \dot{V}_m + X_{16} \dot{W}_m + X_{17} P_m + X_{18} \dot{Q}_m$$

$$\ddot{Q}_m = X_{19} \dot{U}_m + X_{20} V_m + X_{21} W_m + X_{22} \dot{P}_m + X_{23} Q_m$$

By solving the equilibrium equations we obtained the following co-efficients.

$$X_1 = \frac{\beta^2 (A_{11}B_{66} - B_{11}A_{66})}{(A_{11}D_{11} - B_{11}^2)}; \quad X_2 = \frac{A_{11}(\beta^2 D_{66} + KA_{55}) - \beta^2 B_{11}B_{66}}{(A_{11}D_{11} - B_{11}^2)};$$

$$X_3 = \frac{KA_{11}A_{55}}{(A_{11}D_{11} - B_{11}^2)}; \quad X_4 = \frac{\beta[A_{11}(B_{12} + B_{66}) + B_{11}(A_{12} + A_{66})]}{(A_{11}D_{11} - B_{11}^2)};$$

$$X_5 = \frac{\beta[A_{11}(D_{12} + D_{66}) - B_{11}(B_{12} + B_{66})]}{(A_{11}D_{11} - B_{11}^2)}; \quad X_6 = \frac{\beta^2 (A_{22}B_{66} - B_{22}A_{66})}{(B_{66}^2 - A_{66}D_{66})};$$

$$X_7 = \frac{\beta[B_{66}(B_{12} + B_{66}) + A_{66}(D_{12} + D_{66})]}{(B_{66}^2 - A_{66}D_{66})}; \quad X_8 = \frac{-\beta KA_{44}A_{55}}{(B_{66}^2 - A_{66}D_{66})};$$

$$X_9 = \frac{B_{66}(\beta^2 D_{22} + KA_{44}) - \beta^2 B_{22}D_{66}}{(B_{66}^2 - A_{66}D_{66})}; \quad X_{10} = \frac{\beta(B_{12} + B_{66}) + (D_{12} + D_{66})}{(B_{66}^2 - A_{66}D_{66})};$$

$$X_{11} = \frac{\beta A_{44}}{A_{55}}; \quad X_{12} = -1; \quad X_{13} = \frac{\beta^2 A_{44}}{A_{55}};$$

$$X_{14} = \frac{\beta[D_{11}(B_{12} + B_{66}) - B_{11}(D_{12} + D_{66})]}{(A_{11}D_{11} - B_{11}^2)}; \quad X_{15} = \frac{\beta^2 D_{11}D_{66} - B_{11}(\beta^2 D_{66} + KA_{55})}{(A_{11}D_{11} - B_{11}^2)};$$

$$X_{16} = \frac{-KB_{11}A_{55}}{(A_{11}D_{11} - B_{11}^2)}; \quad X_{17} = \frac{\beta[D_{11}(A_{12} + A_{66}) - B_{11}(B_{12} + B_{66})]}{(A_{11}D_{11} - B_{11}^2)};$$

$$X_{18} = \frac{\beta^2 (A_{66}D_{11} - B_{11}B_{66})}{(A_{11}D_{11} - B_{11}^2)}; \quad X_{19} = \frac{\beta^2 B_{22}B_{66} - A_{66}(\beta^2 D_{22} + KA_{44})}{(B_{66}^2 - A_{66}D_{66})};$$

$$X_{20} = \frac{\beta[D_{66}(A_{12} + A_{66}) + B_{66}(B_{12} + B_{66})]}{(B_{66}^2 - A_{66}D_{66})}; \quad X_{21} = \frac{\beta KA_{44}B_{66}}{(B_{66}^2 - A_{66}D_{66})};$$

$$X_{22} = \frac{\beta^2 (B_{22}B_{66} + A_{22}D_{66})}{(B_{66}^2 - A_{66}D_{66})}; \quad X_{23} = \frac{\beta[A_{66}(B_{12} + B_{66}) + B_{66}(A_{12} + A_{66})]}{(B_{66}^2 - A_{66}D_{66})};$$

[A],[B],[D] and [E] matrix for unsymmetric cross-ply laminates as given below

$$\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} = \begin{bmatrix} Q_{11}H_1 & Q_{12}H_2 & 0 \\ Q_{12}H_1 & Q_{22}H_1 & 0 \\ 0 & 0 & Q_{44}H_1 \end{bmatrix}$$

$$\begin{bmatrix} B_{11} & 0 & 0 \\ 0 & -B_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} Q_{11}H_2 & 0 & 0 \\ 0 & -Q_{11}H_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} = \begin{bmatrix} Q_{11}H_3 & Q_{12}H_3 & 0 \\ Q_{12}H_3 & Q_{22}H_3 & 0 \\ 0 & 0 & Q_{44}H_3 \end{bmatrix}$$

$$K\begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} = K\begin{bmatrix} Q_{66}H_1 & 0 \\ 0 & Q_{55}H_1 \end{bmatrix}$$

The second order differential equations are converted into single order

$$S_1 = U_m, S_2 = \dot{U}_m, S_3 = V_m, S_4 = \dot{V}_m, S_5 = W_m$$
$$S_6 = \dot{W}_m, S_7 = P_m, S_8 = \dot{P}_m, S_9 = Q_m, S_{10} = \dot{Q}_m$$

using state space approach the equation may be converted to the form

$$\dot{S} = TS + l$$

Where matrix $T$ is the 10*10 matrix is given as fallows

$$[T] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ X_1 & 0 & 0 & X_2 & 0 & X_3 & X_4 & 0 & 0 & X_5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_6 & X_7 & 0 & X_8 & 0 & 0 & X_9 & X_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_{11} & 0 & 0 & X_{12} & X_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ X_{14} & 0 & 0 & X_{15} & 0 & X_{16} & X_{17} & 0 & 0 & X_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & X_{19} & X_{20} & 0 & X_{21} & 0 & 0 & X_{22} & X_{23} & 0 \end{bmatrix}$$

$l$ is load vector defined as fallows

$$\{l\} = \{0, 0, 0, 0, 0, -q_m / KA_{55}, 0, 0, 0, 0\}^T$$
$$q_m(p) = \frac{2}{b}\int_0^b q(p, q)\sin\frac{m\pi q}{b}dq$$

The solution of $S$ is $S = e^{Tp}\left\{K + \int_{-a/2}^{p} e^{-T\zeta}\, l\, d\zeta\right\}$

Finally the deflection can determined by the using boundary condition on edges $p = \pm a/2$

For simply supported and clamped the boundary conditions as given below the boundary conditions at the edges are $p = \pm a/2$

$$S: v_0 = w_0 = \phi_q = N_p = M_p = 0$$
$$C: u_0 = v_0 = w_0 = \phi_p = \phi_q = 0$$

## IV. NUMERICAL EXAMPLE

Numerical results for various composite plates are presented with different cross-ply lamination schemes under various boundary conditions on two opposite sides while the edges q= (0, b) are simply supported.

It was assumed that the thickness and the material for all the laminate are the same, having the following characteristics:-

$$E_1 = 25E_2, G_{12} = G_{13} = 0.5E_2, G_{23} = 0.2E_2, v_{12} = 0.25$$

The shear correction co-efficient for the first-order theory is taken to be K=5/6. The notations CC, for example, refers to the boundary conditions used on the edges $p = \pm a/2$ ,while the order two edges (i.e., q=0,b) are simply supported.

Figure 2 contain plots of deflection versus side-to-side thickness ratio b/h of two layers and ten layers unsymmetric (0/90/……)even cross-ply laminates (a=b/2) with various boundary conditions. the material properties used are as above .figure 3 contain plot of deflection versus $E_1 / E_2$ for the same load b/h=10 and a=b/2.

The following table contains nondimensionalized center deflection ( $\bar{w}$ ) of unsymmetric cross-ply plates with various boundary conditions.

| Layers | b/h | Theory | Method | SS | CC |
|---|---|---|---|---|---|
| 2 | 5 | FSDT | Exact | 1.758 | 1.257 |
| | | | Matlab | 1.758 | 1.253 |
| | 10 | | Exact | 1.237 | 0.656 |
| | | | Matlab | 1.237 | 0.6584 |
| 10 | 5 | FSDT | Exact | 1.137 | 0.945 |
| | | | Matlab | 1.137 | 0.93 |
| | 10 | | Exact | 0.615 | 0.385 |
| | | | Matlab | 0.615 | 0.3382 |

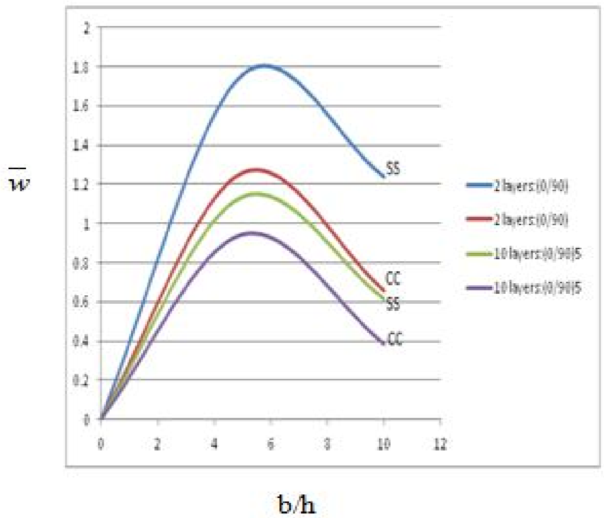


Figure 2: Nondimensionalized center transverse deflection ( $\bar{w}$ ) versus side-to-side thickness ratio b/h for unsymmetric cross-ply laminates.

## V. SUMMARY

Analytical solutions for deflections of unsymmetric rectangular cross-ply laminates with various boundary conditions are presented based on the first-order shear deformation laminate theory. The levy solutions with the states space approach were developed for unsymmetric rectangular cross-ply laminates when two opposite edges are simply supported and other two edges having a variety of boundary conditions of choice.

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