Anti-Fuzzy Graph Coloring

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Abstract- *In this paper, the concept of coloring the anti-fuzzy graph, strong anti-fuzzy graph and complete anti-fuzzy graph are introduced with illustrative examples.*

Keywords- Anti-fuzzy graph,Chromatic number, Edge coloring,Total coloring, Vertex coloring.
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I. INTRODUCTION

Graph coloring dates back to 1852, when Francis Guthrie come up with the four color conjecture. Gary Chartrand and Ping Zhang [3] discussed various colorings of graph and its properties in their book entitled Chromatic Graph Theory. A graph coloring is the assignment of a color to each of the vertices or edges or both in such a way that no two adjacent vertices and incident edges share the same color. Graph coloring has been applied to many real world problems like scheduling, allocation, telecommunications and bioinformatics, etc.

The concept of fuzzy sets and fuzzy relations were introduced by L.A.Zadehin 1965 [15]. A. Rosenfeld who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975 [10]. The concept of chromatic number of fuzzy graph was introduced by Munoz et.al. in 2004 [14]. C. Eslahchi and B.N. Onagh introduced fuzzy graph coloring of fuzzy graph in 2006 [2]. S. Lavanya and R. Sattanathan discussed total fuzzy coloring in 2009 [7]. Anjaly Kishore and M.S. Sunitha discussed chromatic number of fuzzy graph in 2013 [1]. R. Jahir Hussain and K.S. KanzulFathimaconferred fuzzy coloring of fuzzy graph, using strong arcs and dominator coloring of fuzzy graph in 2015 [4, 5, 6].A. NagoorGani and B.FathimaKani deliberated Fuzzy vertex order colouring in 2016 [9].

R. Seethalakshmi and R.B.Gnanajothi introduced Anti-fuzzy graph in 2016 [11] and discussed various properties in 2017 [12, 13].R.Muthuraj and A.Sasireka discussed anti- fuzzy graphs in 2017 [8]. In this paper the attempt has been made to focus the anti-fuzzy graph (AFG), strong anti-fuzzy graph and complete anti-fuzzy graph. In addition to the above graphs, it is also proposed to define the vertex coloring, edge coloring and total coloring of anti-fuzzy graphs interms of a family of anti-fuzzy sets satisfying certain conditions and the chromatic number is the least value of k such that k-coloring exists.

II. PRELIMINARIES

2.1. Definition(L.A.Zadeh [15])

Let X be a non-empty set. Then a fuzzy set A in X (i.e., a fuzzy subset A of X) is characterized by a function of the form $\mu_A: X \to [0,1]$, such a function μ_A is called the membership function and for each $x \in X, \mu_A(x)$ is the degree of membership of x (membership grade of x) in the fuzzy set A. In otherwords, $A = \{(x, \mu_A(x)) / x \in X\}$ where $\mu_A: X \to [0,1]$.

2.2. Definition(A. Rosenfeld [10])

A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma: V \to [0,1]$and $\mu: V \times V \to [0,1]$, where for all $u, v \in V$, we have $\mu(u,v) \leq \sigma(u) \wedge \sigma(v)$.

2.3. Definition(R.Seethalakshmi et.al [11])

Ananti-fuzzy graph (AFG)$G = (\sigma, \mu)$ is a pair of functions $\sigma: V \to [0,1]$and $\mu: V \times V \to [0,1]$, where for all $u, v \in V$, we have $\mu(u,v) \geq \sigma(u) \vee \sigma(v)$.

2.4. Definition (Munoz et.al. [14])

If $G = (V, \mu)$ is such a fuzzy graph where $V = \{1,2,3,\dots,n\}$ and μ is a fuzzy number on the set of all subsets of $V \times V$. Assume $I = A \cup \{0\}$ where $A = \{\alpha_1 < \alpha_2 < \cdots < \alpha_k\}$ is the fundamental set (level set) of G. For each $\alpha \in I, G_\alpha$ denote the crisp graph

$G_\alpha = (V, E_\alpha)$ where $E_\alpha = \{(i,j) / 1 \le i < j \le n, \mu(i,j) \ge \alpha\}$ and $\chi_\alpha = \chi(G_\alpha)$ denote the chromatic number of crisp graph G_α. By this definition the chromatic number of the fuzzy graph G is the fuzzy number $\chi(G) = \{(i, v(i))/i \in X\}$ where $v(i) = \max\{\alpha \in I / i \in A_\alpha\}$ and $A_\alpha = \{1, \dots, \chi_\alpha\}$.

2.5. Definition (Eslahchi and Onagh [2])

A family $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ of fuzzy sets on V is called a k-fuzzy coloring of $G = (V, \sigma, \mu)$ if

a) $\vee\ \Gamma = \sigma$,

b) $\gamma_i \wedge \gamma_j = 0$,

c) For every strong edge xy of G, min $\{\gamma_i(x), \gamma_i(y)\} = 0 (1 \le i \le k)$

2.6. Definition (S. Lavanya and R. Sattanathan [7])

A family $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ of fuzzy sets on $V \cup E$ is called a k-fuzzy total coloring of $G = (V, \sigma, \mu)$ if

a) $max_i \gamma_i(v) = \sigma(v)$ for all $v \in V$ and $max_i \gamma_i(uv) = \mu(uv)$ for all edge $uv \in E$

b) $\gamma_i \wedge \gamma_j = 0$,

c) For every adjacent vertices u, v of min $\{\gamma_i(u), \gamma_i(v)\} = 0$ and for every incident edges

$$\min \left\{ \begin{array}{c} \gamma_i(v_j, v_k) \ / \ v_j, v_k \ are\ set\ of\ incident \\ edges\ from\ the\ vertex\ v_j, \\ j = 1, \dots, |v| \end{array} \right\}$$

2.7.Definition(R.Seethalakshmi et.al [12])

Ananti-fuzzy graph, $G = (\sigma, \mu)$is said to be strong anti-fuzzy graph if $\mu(v_i v_j) = \max(\sigma(v_i), \sigma(v_j))$, for all $(v_i, v_j) \in E$.

2.8. Definition(R.Seethalakshmi et.al [12])

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Ananti-fuzzy graph, $G = (\sigma, \mu)$is said to be complete anti-fuzzy graph if $\mu(v_i v_j) = \max(\sigma(v_i), \sigma(v_j))$,for every $v_i, v_j \in V$.

III. ANTI-FUZZY GRAPH COLORING

3.1. Definition

1. The arc (u, v) in anti-fuzzy graph G is said to be a strong arc if $\mu(u, v)$is greater than or equal to the strength of connectedness between $u$and v.
2. Two vertices u and v in anti-fuzzy graph G are called adjacent if (u, v) is strong arc in G otherwise weakly adjacent.
3. If two distinct edges (u, v) and (v, w) in anti-fuzzy graph G are incident with a common vertex v, then they are called incident edges.

3.2. Definition (Vertex coloring)

A family $C = \{c_1, \dots, c_k\}$ of anti-fuzzy sets on a set V is called a k-vertex coloring of $G = (V, \sigma, \mu)$ if

(i) $\vee\ c_i(x) = \sigma(x)$, for all $x \in V$

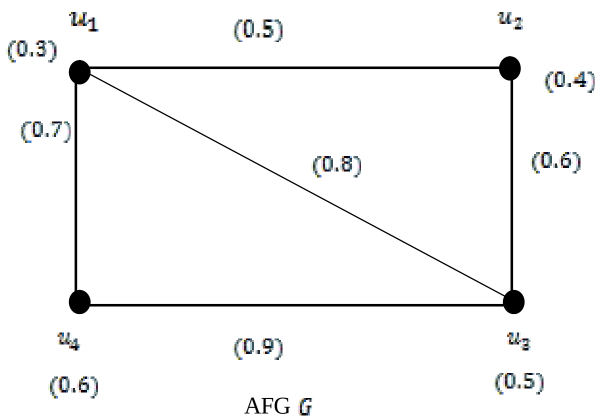
(ii) $c_i \wedge c_j = 0$

(iii) For every strong edge xy of G, $\min\{c_i(\sigma(x)), c_i(\sigma(y))\} = 0$, $(1 \le i \le k)$

The least value of k for which the G has a k-vertex coloring denoted by $\chi(G)$, is called the chromatic number of the anti-fuzzy graph G.

3.3. Example (Anti-fuzzy graph vertex coloring)

Consider the anti-fuzzy graph $G = (\sigma, \mu)$ in figure 3.1, with four vertices and five edges.



AFG G

Figure 3.1

Let $C = \{c_1, c_2, c_3\}$ be a family of anti-fuzzy sets defined on V as follows

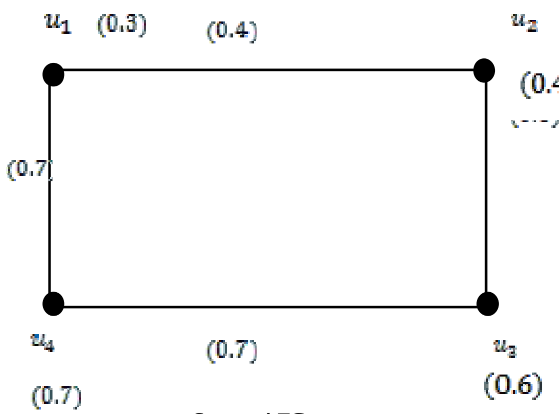
$$c_1(u_i) = \begin{cases} (0.3), & i = 1 \\ (0), & otherwise \end{cases} \quad c_2(u_i) = \begin{cases} (0.4), & i = 2 \\ (0.6), & i = 4 \\ (0), & otherwise \end{cases}$$

$$c_3(u_i) = \begin{cases} (0.5), & i = 3 \\ (0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2, c_3\}$ satisfies our definition of vertex coloring of anti-fuzzy graph. We find that any family of anti-fuzzy sets having less than three members could not satisfy our definition. Hence in this case the chromatic number $\chi(G)$ is 3.

3.3. **Example** (strong anti-fuzzy graph vertex coloring)
3.4.

Consider the strong anti-fuzzy graph $G = (\sigma, \mu)$ in figure 3.2, with four vertices and four edges.



Strong AFG G

Figure 3.2

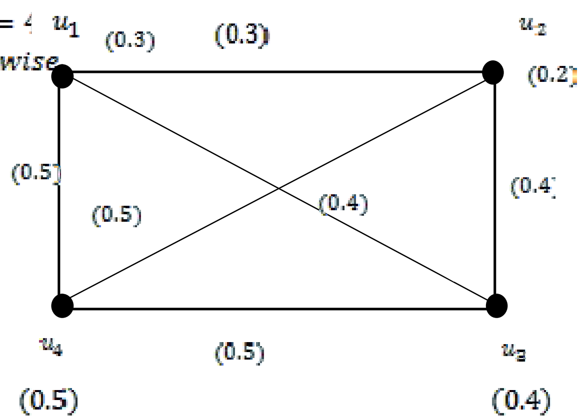
Let $C = \{c_1, c_2\}$ be a family of anti-fuzzy sets defined on V as follows

$$c_1(u_i) = \begin{cases} (0.3), & i = 1 \\ (0.6), & i = 3 \\ (0), & otherwise \end{cases} \quad c_2(u_i) = \begin{cases} (0.4), & i = 2 \\ (0.7), & i = 4 \\ (0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2\}$ satisfies our definition of vertex coloring of strong anti- fuzzy graph. We find that any family of anti-fuzzy sets having less than two members could not satisfy our definition. Hence in this case the chromatic number $\chi(G)$ is 2.

3.5. Example (complete anti-fuzzy graph vertex coloring)

Consider the complete anti-fuzzy graph $G = (\sigma, \mu)$ in figure 3.3, with four vertices and six edges.



Complete AFG G

Figure 3.3

Let $C = \{c_1, c_2, c_3, c_4\}$ be a family of anti-fuzzy sets defined on V as follows

$$c_1(u_i) = \begin{cases} (0.3), & i = 1 \\ (0), & otherwise \end{cases} \quad c_2(u_i) = \begin{cases} (0.2), & i = 2 \\ (0), & otherwise \end{cases}$$

$$c_3(u_i) = \begin{cases} (0.4), & i = 3 \\ (0), & otherwise \end{cases} \quad c_4(u_i) = \begin{cases} (0.5), & i = 4 \\ (0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2, c_3, c_4\}$ satisfies our definition of vertex coloring of complete anti- fuzzy graph. We find that any family of anti-fuzzy sets having less than four members could not satisfy our definition. Hence in this case the chromatic number $\chi(G)$ is 4.

3.6. Bound for chromatic number of AFG

3.6.1. Proposition

For any AFG, the chromatic number $\chi(G) \leq \Delta(G)+1$ where $\Delta(G)$ is the maximum number of edges incident to a vertex of G.

3.6.2. Proposition

The chromatic number of complete AFG is n where n is the number of vertices of G, i.e., $\chi(G)=n$.

(Since $\Delta(G)=n-1$ is the maximum vertex degree of the complete AFG G, chromatic number $\chi(G)=\Delta(G)+1$)

Proof

Since $\mu(x_i y_j)=\max(\sigma(x_i),\sigma(y_j))$ for every $x_i, y_j \in V$. Every pair of vertices are adjacent and degree of each vertex is $n-1$. By (iii) of definition [3.2], $\min\{c_i(\sigma(x)), c_i(\sigma(y))\}=0$, for adjacent vertices x, y. Since all vertices are adjacent, every member of the family defining anti-fuzzy graph coloring have value for only one vertex and (0) for all other vertices. By (ii) of definition [3.2], $c_i \wedge c_j = 0$, so $\vee\, c_i(x_i)=\sigma(x_i)$ for all $x_i \in V i=1,2,\ldots,n$ where $n=|V|$. Thus, $\chi(G)=n$.

3.7. Definition (Edge coloring)

A family $C=\{c_1,\ldots,c_k\}$ of anti-fuzzy sets on E is called a k-edge coloring of $G=(V,\sigma,\mu)$ if

a) $\vee\, c_i(xy)=\mu(xy)$, for all edge $xy \in E$
b) $c_i \wedge c_j = 0$
c) For every incident edges xy on vertex $x \in V$ of G, min $\{c_i(\mu(xy))\}=0$, $(1 \leq i \leq k)$

The least value of k for which the G has a k- edge coloring denoted by $\chi'(G)$, is called the edge chromatic number of the anti-fuzzy graph G.

3.8. Example (Anti-fuzzy graph edge coloring)

Consider the AFG $G=(\sigma,\mu)$, in Example 3.3. (figure 3.1)

Let $c=\{c_1,c_2,c_3\}$ be a family of anti-fuzzy sets defined on E as follows

$$c_1(u_iu_j)=\begin{cases}(0.5), & ij=12\\(0.9), & ij=34\\(0), & otherwise\end{cases}\quad c_2(u_iu_j)=\begin{cases}(0.7), & ij=14\\(0.6), & ij=23\\(0), & otherwise\end{cases}$$

$$c_3(u_iu_j)=\begin{cases}(0.8), & ij=13\\(0), & otherwise\end{cases}$$

Hence the family $C=\{c_1,c_2,c_3\}$ satisfies our definition of edge coloring of anti-fuzzy graph. We find that any family of anti-fuzzy sets having less than three members could not satisfy our definition. Hence in this case the edge chromatic number $\chi'(G)$ is 3.

3.9. Example (strong anti- fuzzy graph edge coloring)

Consider the strong AFG $G=(\sigma,\mu)$, in Example 3.4. (figure 3.2)

Let $C=\{c_1,c_2\}$ be a family of anti-fuzzy sets defined on E as follows

$$c_1(u_iu_j)=\begin{cases}(0.4), & ij=12\\(0.7), & ij=34\\(0), & otherwise\end{cases}\quad c_2(u_iu_j)=\begin{cases}(0.7), & ij=14\\(0.6), & ij=23\\(0), & otherwise\end{cases}$$

Hence the family $C=\{c_1,c_2\}$ satisfies our definition of edge coloring of strong anti-fuzzy graph. We find that any family of anti-fuzzy sets having less than two members could not satisfy our definition. Hence in this case the edge chromatic number $\chi'(G)$ is 2.

3.10. Example (complete anti-fuzzy graph edge coloring)

Consider the complete AFG $G=(\sigma,\mu)$, in Example 3.5. (figure 3.3)

Let $C=\{c_1,c_2,c_3\}$ be a family of anti-fuzzy sets defined on E as follows

$$c_1(u_iu_j)=\begin{cases}(0.3), & ij=12\\(0.5), & ij=34\\(0), & otherwise\end{cases}\quad c_2(u_iu_j)=\begin{cases}(0.4), & ij=23\\(0.5), & ij=41\\(0), & otherwise\end{cases}$$

$$c_3(u_iu_j)=\begin{cases}(0.4), & ij=13\\(0.5), & ij=24\\(0), & otherwise\end{cases}$$

Hence the family $C=\{c_1,c_2,c_3\}$ satisfies our definition of edge coloring of complete anti- fuzzy graph. We find that any family of anti-fuzzy sets having less than three members could not satisfy our definition. Hence in this case the edge chromatic number $\chi'(G)$ is 3.

3.11. Bound for edge chromatic number of AFG

3.11.1. Proposition

For any AFG, the edge chromatic number $\chi'(G) \le \Delta(G) + 1$ where $\Delta(G)$ is the maximum number of edges incident to a vertexof G.

3.11.2. Proposition

The edge chromatic number of complete AFG on n vertices is n if n is odd and $n-1$ if n is even.

3.12. Definition(Total coloring)

A family $C = \{c_1, \dots, c_k\}$ of anti-fuzzy sets on $V \cup E$ is called a k- total coloring of $G = (V, \sigma, \mu)$ if

a) $\vee\ c_i(x) = \sigma(x)$, for all $x \in V$ and $\vee\ c_i(xy) = \mu(xy)$, for all edge $xy \in E$

b) $c_i \wedge c_j = 0$,

c) For every strong edge xy of G, $\min\{c_i(\sigma(x)), c_i(\sigma(y))\} = 0$. And for every incident edges xy on vertex $x \in V$ of G, min $\{c_i(\mu\ \ (xy))\} = 0$, $(1 \le i \le k)$.

The least value of k for which the G has a k- total coloring denoted by $\chi^T(G)$, is called the total chromatic number of the anti- fuzzy graph G.

3.13. Example (Anti-fuzzy graph total coloring)

Consider the AFG $G = (\sigma, \mu)$, in Example 3.3. (figure 3.1)

Let $C = \{c_1, c_2, c_3, c_4\}$ be a family of anti-fuzzy sets defined on $V \cup E$ as follows

$$c_1(u_i) = \begin{cases} (0.3), & i = 1 \\ (0), & otherwise \end{cases} \quad c_2(u_i) = \begin{cases} (0.4), & i = 2 \\ (0.6), & i = 4 \\ (0), & otherwise \end{cases}$$

$$c_3(u_i) = \begin{cases} (0.5), & i = 3 \\ (0), & otherwise \end{cases}$$

$$c_1(u_iu_j) = \begin{cases} (0.6), & ij = 23 \\ (0), & otherwise \end{cases} \quad c_2(u_iu_j) = \begin{cases} (0.8), & ij = 13 \\ (0), & otherwise \end{cases}$$

$$c_3(u_iu_j) = \begin{cases} (0.7), & ij = 41 \\ (0), & otherwise \end{cases} \quad c_4(u_iu_j) = \begin{cases} (0.5), & ij = 12 \\ (0.9), & ij = 34 \\ (0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2, c_3, c_4\}$ satisfies our definition of total coloring of anti-fuzzy graph. We find that any family of anti-fuzzy sets having less than four members could not satisfy our definition. Hence in this case the total chromatic number $\chi^T(G)$ is 4.

3.14. Example (strong anti-fuzzy graph total coloring)

Consider the strong AFG $G = (\sigma, \mu)$, in Example 3.4. (figure 3.2)

Let $C = \{c_1, c_2, c_3, c_4\}$ be a family of anti-fuzzy sets defined on $V \cup E$ as follows

$$c_1(u_i) = \begin{cases} (0.3), & i = 1 \\ (0.6), & i = 3 \\ (0), & otherwise \end{cases} \quad c_2(u_i) = \begin{cases} (0.4), & i = 2 \\ (0.7), & i = 4 \\ (0), & otherwise \end{cases}$$

$$c_3(u_iu_j) = \begin{cases} (0.4), & ij = 12 \\ (0.7), & ij = 34 \\ (0), & otherwise \end{cases} \quad c_4(u_iu_j) = \begin{cases} (0.6), & ij = 23 \\ (0.7), & ij = 41 \\ (0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2, c_3, c_4\}$ satisfies our definition of total coloring of strong anti- fuzzy graph. We find that any family of anti-fuzzy sets having less than four members could not satisfy our definition. Hence in this case the total chromatic number $\chi^T(G)$ is 4.

3.15. Example (complete anti-fuzzy graph total coloring)

Consider the completeAFG $G = (\sigma, \mu)$, in Example 3.5. (figure 3.3)

Let $C = \{c_1, c_2, c_3, c_4, c_5\}$ be a family of anti-fuzzy sets defined on $V \cup E$ as follows

$$c_1(u_i) = \begin{cases} (0.3), & i = 1 \\ (0), & otherwise \end{cases} \quad c_2(u_i) = \begin{cases} (0.2), & i = 2 \\ (0), & otherwise \end{cases}$$

$$c_3(u_i) = \begin{cases} (0.4), & i = 3 \\ (0), & otherwise \end{cases} \quad c_4(u_i) = \begin{cases} (0.5), & i = 4 \\ (0), & otherwise \end{cases}$$

$c_1(u_iu_j) = \begin{cases} (0.5), & ij = 24 \\ (0), & otherwise \end{cases}$ $c_2(u_iu_j) = \begin{cases} (0.4), & ij = 13 \\ (0), & otherwise \end{cases}$

$c_3(u_iu_j) = \begin{cases} (0.5), & ij = 41 \\ (0), & otherwise \end{cases}$ $c_4(u_iu_j) = \begin{cases} (0.4) & ij = 23 \\ (0), & otherwise \end{cases}$

$$c_5(u_iu_j) = \begin{cases} (0.3), & ij = 12 \\ (0.5), & ij = 34 \\ (0), & otherwise \end{cases}$$

Hence the family $C = \{c_1, c_2, c_3, c_4, c_5\}$ satisfies our definition of total coloring of complete anti-fuzzy graph. We find that any family of anti-fuzzy sets having less than five members could not satisfy our definition. Hence in this case the total chromatic number $\chi^T(G)$ is 5.

3.16. Bound for total chromatic number of AFG

3.16.1. Proposition

Any AFG $G = (V, E)$ can be totally colored using at most $\Delta + 2$ colors. That is, the total chromatic number of AFG $\chi^T(G) \leq \Delta(G) + 2$ where Δ is the maximum number of edges incident from a vertex in G.

IV. CONCLUSION

This paper tried to define the vertex, edge and total coloring for anti-fuzzy graph, strong anti-fuzzy graph and complete anti-fuzzy graph with elucidative examples and achieved the chromatic number as a crisp number. Moreover, it has also found bounds for that chromatic numbers.

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