The Circular Chromatic Number of A Digraph

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I. INTRODUCTION

We shall consistently use *D* to denote digraphs and *G* to denote (simple, undirected) graphs. An edge of *G* denoted by *uv* represents the edge joining the vertices *u* and *v*. In a digraph, the arc *uv* has initial vertex *u* and terminal vertex *v*. All graphs and digraphs are simple, but we allow oppositely oriented arcs *uv* and *vu* to belong to the arc set of a digraph.

For a positive real number *p*, denote by $S_p \subset R^2$ the circle with perimeter *p* (hence with radius $p/2\pi$) centered at the origin of R^2. We can identify S_p with the set R/pZ in the obvioius way. For $x, y \in S_p$, let us denote by $S_p(x,y)$ the arc on S_p from *x* to *y* in the clockwise direction and let $d(x,y)$ denote the length of this arc. The set R/pZ can also be identified with the real interval $[0,p]$, where the distance function $d(x,y)$ can be expressed as

$$d(x,y) = \begin{cases} y - x & \text{if } x \text{ precedes } y \text{ on } [0,p) \\ p + y - x, & \text{otherwise} \end{cases}$$

Proposition: 4.1.1

$$\chi(D) - 1 < \chi_c(D) \le \chi(D)$$

Proof:

Clearly, a *k*-coloring of *D* determines a weak circular *k*-coloring of *D*.

This yields the second inequality.

Let $p = \chi_c(D)$, $k = [p]$ and $\varepsilon = p/2n$, where *n* is the order of *D*, and let *c* be a circular $(p+\varepsilon)$ - coloring.

Then $S_{p+\epsilon}$ can be written as the union of $k+1$ (disjoint) arcs $A_0, A_1, \cdots, A_k$ each of length less than 1, and such that $c^{-1}(A_0) = \phi$ .For $i = 1, \cdots, k$.

Let $V_i = c^{-1}(A_i)$

Clearly, each V_i is acyclic, and the partition of $V(D)$ into these acyclic sets is a *k*-coloring of *D*. This verify the first inequality.

Hence the proof.

Let the relations in proposition 4.4.1 also hold between the chromatic and circular chromatic numbers of undirected graphs .

This result, together with the fact that $\chi_c(C_{2k+1}) = 2 + 1/k$, was one motivation for introducing the circular chromatic number. The fact that χ_c for odd cycles monotonically decreases towards 2 has an even more natural digraph counterpart.

Namely, for directed cycles $\vec{C}_n$, the circular chromatic number monotonically approaches 1 as the length *n* increases

$$\chi_c\left(\vec{C}_n\right) = 1 + \frac{1}{n-1}$$

This result is a special case of proposition 4.1.2 below.

Let $C(k,d)$ be the undirected graph with vertex set $\{0, \cdots, k-1\}$ in which distinct vertices i, j are adjacent if and only if $d \le |i - j| \le k - d$.

If $k \ge 2d$, then this graph has circular chromatic number k/d See [7] or [1]. Here we define a directed analogue of $C(k,d)$: let $\overrightarrow{C}(k,d)$ be the digraph with vertex set $V\left(\overrightarrow{C}(k,d)\right)=\{0,\cdots,k-1\}$ whose arcs emanate from a given vertex $i\in V\left(\overrightarrow{C}(k,d)\right)$ to the vertices $i+d$, $i+d+1,\cdots,i+k-1$ with arithmetic modulo *k*. We display $\overrightarrow{C}(7,3)$ in figure 1. Notice that $\overrightarrow{C}(n,n-1)\cong\overrightarrow{C}_n$.

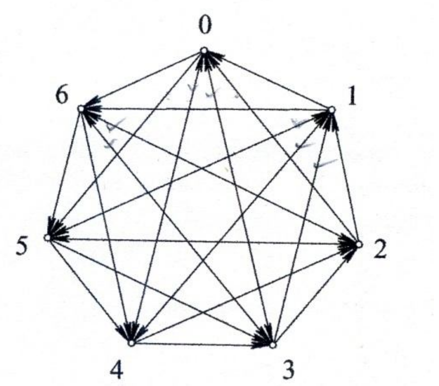


Figure: The digraph $\overrightarrow{C}(7,3)$

As noted above, $\chi_c(D)$ is always rational and at least 1.

Proposition: 4.1.2

If *k* and *d* are positive integers $k\geq d$, then $\chi_c\left(\overrightarrow{C}_n(k,d)\right)=\dfrac{k}{d}$

In particular, for every rational number $p\geq 1$, there exists a digraph with circular chromatic number *p*.

Proof: Let $p=k/d$. It is easy to see that, $c:V\left(\overrightarrow{C}(k,d)\right)\to R/pZ$ defined by $c(i):=i/d$ is a circular *p*-coloring.

Therefore $\chi_c\left(\overrightarrow{C}(k,d)\right)\leq k/d$. If $k\geq 2d$, then $\overrightarrow{C}(k,d)$ contains $C(k,d)$ as a subdigraph (where each edge of $C(k,d)$ is replaced by a pair of oppositely oriented arcs).

Thus, in this case $\chi_c\left(\overrightarrow{C}(k,d)\right)\geq\chi_c(C(k,d))$ which by [1] is k/d.

It remains to consider the case $d<k<2d$. Suppose, for a contradiction, that $\overrightarrow{C}(k,d)$ admits a circular *q*-coloring *c* with $q<k/d$. We may assume that $C(0)=0$. Let $d_{ij}=d(c(i),c(j))$ Then $\sum_{i=0}^{k-1}d_{i,i+1}=lq$ for some positive integer *l*. Since $\overrightarrow{C}(k,d)$ contains each arc $i0$, for $1\leq i\leq k-d$, we have $0<c(i)\leq q-1$ for each such *i*.

Since $q-1<1$ it follows that $c(i)<c(j)$ Whenever $i<j$ and $i,j\in\{1,\cdots,k-d\}$.

Thus $\sum_{i=0}^{k-d-1}d_{i,i+1}\leq q-1$ and by symmetry, the same bound holds for the sum of any $k-d$ consecutive values $d_{i,i+1}$. Summing the resulting *k* inequalities, we obtain $(k-d)lq=(k-d)\sum_{i=0}^{k-1}d_{i,i+1}\leq k(q-1)$, which shows that $l\leq k(q-1)/(k-d)<1$.

Contradicting $l\in Z^+$.

Therefore $\chi_c\left(\vec{C}(k,d)\right)\geq k/d$.

Hence the proof.

Definition: 4.1.3

An cyclic homomorphism of a digraph D into a digraph D' is a mapping $\phi: V(D) \to V(D')$ such that,

i) for every arc $uv \in E(D)$, either $\phi(u) = \phi(v)$ or $\phi(u)\phi(v)$ is an arc of D'.

ii) For every vertex $v \in V(D')$ the subgraph of D induced on $\phi'(v)$ is acyclic.

Proposition: 4.1.4

A digraph D has circular chromatic number at most k/d if and only if there exists an acyclic homomorphism $D \to \vec{C}(k,d)$.

Proof:

Let $p = k/d$.Suppose that $\chi_c(D) \le p$ and let $c: V(D) \to \mathbb{R}/pZ$ by a weak circular p-coloring of D.

Then $\phi: V(D) \to V\left(\vec{C}(k,d)\right)$ defined by $\phi(u) = i$ where $c(u) \in [i/d,\ (i+1)/d]$ is an acyclic homomorphism from D into $\vec{C}(k,d)$.Conversely,if $\phi: V(D) \to V\left(\vec{C}(k,d)\right)$ is an acyclic homomorphism then $c(u) := \phi(u)/d$ defines a weak circular p-coloring of D.

Hence the proof.

4.2 TIGHT CYCLES AND DEGENERACY

Let c be a weak circular p-coloring of a digraph D. A cycle $C = v_1v_2, \cdots, v_kv_1$ in the underlying graph of D is tight (with respect to c) if for every edge v_iv_{i+1} of C $(i = 1, \cdots, k)$ with induces modulo k we have $d(c(v_i), c(v_{i+1})) = 1$.

Whenever v_iv_{i+1} is an arc of D and $c(v_i) = c(v_{i+1})$. Otherwise,

If C is a tight cycle, then its weight $a(C)$ is the number of edges v_iv_{i+1} that are also arcs of D.

Clearly, the weight of a tight cycle is an integral multiple of p: We call the value $w(C) = a(C)/p$ the winding number of C.

Proposition: 4.2.1

If D is weakly k- degenerate, then $\chi(D) \le k+1$.

Proof:

Let $v_1, \cdots, v_n$ be the vertices of D enumerated so that for $i = 1, \cdots, n$ the vertex v_i has either indegree or outdegree at most k in the induced subdigraph $D_i = D[\{v_1, \cdots, v_i\}]$. Define $A_0, \cdots, A_k$ as follows. Start with empty sets for $i = 1, \cdots, n$ there is a set A_j with $j = j(i)$ such that A_j contains either no out neighbors or no in neighbors of v_i in D_i.Now put v_i in A_j.

Suppose that one of the resulting sets A_j contains a cycle C.

If v_i is the vertex on C with largest index i, then v_i has an in and an out neighbour among the other vertices on C, which is impossible by the construction of the sets $A_0, \cdots, A_k$.

Therefore, the partition into these sets determines a $(k+1)$ coloring of D.

Proposition: 4.2.2

For every non-negative integer k, there exists a $2k$-degenerate graph G_k and a weakly k-degenerate orientation D_k of G_k with $\chi(D_k) = k+1$.

Proof:

The digraphs D_k are constructed inductively for $k \ge 0$.

We let D_0 be the graph K_1 and having constructed D_{k-1}. Obtain D_k as follows

If $V(D_{k-1}) = V_1 \cup \cdots \cup V_k$ is a *k*-coloring of D_{k-1}, let *r* be the number of color classes V_i whose induced subdigraph has at least one arc.

We say that this *K*-coloring has strength *r*. Next, we define weakly *k*-degenerate digraph D_k^r for $r = 0, 1, \cdots, k+1$ such that every *k*-coloring of D_k^r has strength at least *r*.

In particular, D_k^{k+1} has no *k*-colorings and we shall take this graph as D_k.

Let $D_k^0 = D_{k-1}$

Inductively, having constructed D_k^r. Where $0 \le r \le k$. We consider all *k*-colorings of D_k^r strength *r*.

For every such *k*-coloring with color classes $V_1, \cdots, V_k$. We add a new vertex *v* to D_k^r which has precisely one outgoing arc to each color class and at most one incoming arc from each color class and at most one incoming arc from each color class.

If V_i has no arcs, then *v* is joined to an arbitrary vertex in V_i.

If V_i has an arc $v_i u_i$, then we add arcs $v v_i$ and $u_i v$.

It is clear that the given *k*-coloring of D_k^r cannot be extended to a *k*-coloring of $D_k^r + v$ of strength at most *r*.

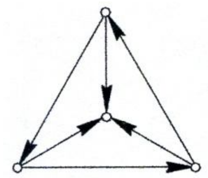
The digraph obtained by adding vertices for all *k*-colorings of D_k^r of strength *r* is the new digraph D_k^{r+1}.

It is clear by construction that all digraphs D_k^r and hence also D_k are weakly *k*-degenerate and also that the underlying graph G_k and D_k is $2k$-degenerate.

Hence the proof.

Corollary: 4.2.3 Let *G* be a connected graph with maximum degree at most 4. If *D* is an orientation of *G* which is not Eulerian, then $\chi(G) \le 2$.

There are digraphs of maximum degree at most 3 with circular chromatic number 2. Clearly, if *D* contains a 2-cycle, then $\chi_c(D) = 2$.



A 2-critical digraph for χ_c

Another example is the cubic digraph in Figure (2) To see this, let 1, 2, 3 be the consecutive vertices of the oriented 3-cycle and
let 4 be the central vertex (which has indegree 3). Assume that $p < 2$ and let $c : V \to \mathbb{R} / pZ$ be a circular coloring with $c(1) = 0$. Then $c(2) \in [1, p)$ and $c(3) \in [c(2) + 1 - p, c(2)) \cap (0, p-1]$.
Hence $c(1), c(2), c(3)$ partition $[0, p]$ into 3 intervals, each of length less than 1. This leaves no room for $c(4)$.

Definition: 4.3

The circular chromatic number $\chi_c(G)$ of the edge-weighed graph *G* is the infimum of all real numbers *p* for which there exists a circular *p*-coloring of *G*.

Tight Edges: 4.3.1

First, we shall show that the infimum in the definition of the circular chromatic number is attained. It will be shown that for every weighted graph *G* of order *n*, there exists a circular *p*-coloring for $p = \chi_c(G)$, and that $\chi_c(G)$ can be expressed as an integer fraction with denominator smaller than *n* of a sum at most *n* edge weights. This implies, in particular, that $\chi_c(G)$ is a rational number if all edge-weights are rational. Let *c* be a circular *p*-coloring of *G*. A directed edge (u, v) is said to be tight if $d(c(u), c(v)) = a_{uv}$. A cycle $C = v_1 v_2, \cdots, v_k v_1$ is tight if the directed edges $(v_1, v_2) \cdots (v_{k-1}, v_k)$ and (v_k, v_1) are all tight.

If $k = 2$ and the edges (v_1, v_2) and (v_2, v_1) are both tight, then we also consider the 2-cycle $v_1 v_2 v_1$ to be a tight cycle. If *C* is tight cycle, then the weight of *C*.

$a(C) = a_{v_1 v_2} + \cdots + a_{v_{k-1} v_k} + a_{v_k v_1}$

is an integer multiple of p and the number $w(C)=a(C)/p$ is called the winding number of *C*.

Lemma: 4.3.2

If $p_0=\chi_c(G)$, then there is a circular p_0- coloring of *G* which has a tight cycle.

Proof:

We may assume that *G* is connected suppose that $p_1 \geq p_0$ and that there is a circular p_1-coloring of *G*.

Let $p \leq p_1$ be a real number such that *G* has a circular *p*-coloring *c* with maximum number of tight edges.

Let v_0 be a vertex of *G* and let $v_0=\{v_0\}$. For $i=1,2,\cdots$. Let $V_i=\{v\in V \mid \text{there exists } u\in V_{i-1} \text{ such that } (u,v) \text{ is tight}\}$ .If $V_0\cup\cdots\cup V_{i-1}\neq V$ and $V_i=\phi$, we can shift the colors of $V\setminus(V_0\cup\cdots\cup V_{i-1})$ counter clockwise until a new tight edge occurs.

By the maximality of *c*, this does not happen. Consequently, for each $v\in V$ there is a path from v_0 to v consisting only of tight edges. Suppose that there is no tight cycle for $v\in V$. Let $l(v)$ be the maximum of $a(P)$ taken over all directed walks *P* from v_0 to *v* which consists of tight edges only. Since there are no tight cycles, the values $l(v)$ are finite. By the definition of $l(v)$, if $l(u)>l(v)$, then the edge *uv* is not tight. This implies that for $p'=p-\in$, where $\in>0$ is small enough, the mapping $c'(v)=l(v) \bmod p' \in \mathbb{R}/p'Z$ determines a circular p'-coloring of *G*.

By increasing the value of $\in$ as much as possible, a new tight edges occurs.

This contradiction to the maximality of *c* shows that there exists a tight cycle.

Clearly, if the cycle *C* is tight, then $p=a(c)/w(c)$ winding number of *c* is bounded by the number of edges in *C*.

Therefore, the same cycle can be tight for at most *n* distinct values of *p*. Since there are only finitely many cycles of *G*, this easily implies the statement of the lemma.

Corollary: 4.3.3

For every forest *F* with at least one edge, $\chi_c(F)=\max\{a_{uv}+a_{vu} \mid u,v\in V\}$

Let *C* be a tight cycle with respect to a circular *p*-coloring. Then $p=a(C)/w(C)$.

If the edge weights are symmetric, then each edge weight is at most $p/2$.

Therefore, $a(C)\leq np/2$, so the winding number is atmost $n/2$.

In the non symmetric case, the winding number is atmost $n-1$.

4.4 UPPER BOUNDS

Let $G=(V,A)$ be a weighted graph for $v\in V$, let $d_G^+(v)=\sum_{u\in V} a_{uv}$ and $d_G^-(v)=\sum_{u\in V} a_{vu}$ and let $D_G(v)=d_G^+ + d_G^-(v)$. The graph *G* is said to be weakly *p*-degenerate if every subgraph *H* of *G* contains a vertex *v* with $D_H(v)\leq p$.

Proposition: 4.4.1

The graph G_n is $(n-1)$-generate. In *n* is odd, then $\chi_c(G_n)=2n-4+\frac{4}{n+1}$.

Proof:

It is clear by construction that G_n is $(n-1)$ degenerate suppose now that *n* is odd. Let $p_0=2n-4+\frac{4}{n+1}=nr$, where $r = 2 - \frac{4}{n+1}$. Let $v_1, \cdots, v_n$ be the vertices of $k_n \subset G_n$.By setting $c(v_i) = (i-1)r \in \mathbb{R} / p_0 Z$, a circular p_0- coloring of K_n is obtained which can be extended to a circular p_0-coloring of G_n. Therefore, $\chi_c(G_n) \leq p_0$.

Suppose now that $p < p_0$ and suppose that there is a circular *p*-coloring *c* of G_n. Let $x_i = c(v_i)$ We may assume that the cyclic order of these colors on S_p is $x_1, \cdots, x_n$. For $x \in S_p$. Let $\bar{x}$ be the point of S_p which lies diametrically opposite *x* on thecircle S_p.Let $r_i = d(x_i, x_{i+1}),\ i = 1, \cdots, n-1$ and let $r_n = d(x_n, x_1)$.

Let α be the minimum distance of a point x_j from some $\bar{x}_i, i, j \in \{1, \cdots, n\}$.

Since the color $c\left(t_{v_i v_j}\right)$ has distance at least *k* from x_i and from x_j. It is necessary that $p/2 + \alpha \geq 2k$.

This implies, $\alpha \geq 2k - \frac{p}{2} > 1 - \frac{2}{n+1}$ (1)

If the opposite segment of $S_p(x_i, x_{i+1})$ contains some point $x_j (1 \leq j \leq n)$, then we say that *i* is normal. Otherwise *i* is said to be abnormal.

If *i* is normal, then (1) implies that $r_i > 2 - \frac{4}{n+1}$ (2)If *i* is abnormal, let *j* be the index such that $S_p(x_j, x_{j+1})$ contains $\bar{x}_i$. Then *j* is normal. This shows that there exists a normal index, and we shall assume that *n* is normal. Let $i, i+1, \cdots, i+k \ (k \geq 0)$ be a maximal subsequence of $1, \cdots, n$ such th $i, \cdots, i+k$ are all abnormal.

Let $j(1 \leq j \leq n)$ be the index such that the segment $S_p(x_j, x_{j+1})$ (index $j+1$ modulo *n*) contains $\bar{x}_i$. Then $S_p(x_j, x_{j+1})$ also contains $\bar{x}_{i+1}, \cdots, \bar{x}_{i+k+1}$.

This fact and (1) imply that $r_j > 2\left(1 - \frac{2}{n+1}\right) + r_i + \cdots + r_{i+k}$

Recall that $r_l \geq 1 \quad (1 \leq l \leq n)$.Consequently, $r_i + \cdots + r_{i+k} + r_j > 2(k+2) - \frac{4}{n+1}$.

Since every *j* appears at most once opposite some maximal abnormal sequence $i, \cdots, i+k$ and since every normal *i* satisfies (2).We get $p = r_1 + r_2 + \cdots + r_n > 2n - \frac{4n}{n+1} = p_0$ a contradiction. This shows that $\chi_c(G_n) = p_0$.

Hence the proof.

Theorem: 4.4.2

Let $\Delta^+(G) = \max\left\{d_G^+(v) + a_{vu} \mid u, v \in V\right\}$. Then $\chi_c(G) \leq \Delta^+(G)$.

Proof:

Let us assume that all edge weights are integers. Let $v_1, \cdots, v_n$ be the vertices of *G*. We assign them colors $c(v_i) \in N(i, \cdots, n)$ by applying the following "Greedy" algorithm.

For consecutive values of $\alpha = 0, 1, 2, \cdots$ traverse all vertices $v_1, \cdots, v_n$ and assign color $c(V_i) = \alpha$ to every uncolored vertex v_i such that for every vertex v_j that already received a color $c(v_j)$, $\alpha - c(v_j) \geq a_{v_j v_i}$

We claim that $c(v_i) \leq d_G^+(v_i),\ i = 1, \cdots, n$. Suppose that v_i has not been colored for $\alpha = 0, 1, \cdots, d$.

Then for each such α, there was a vertex $v_j(\alpha)$ such that $\alpha - c\left(v_{j(\alpha)}\right) < \alpha_{v_{j(\alpha)}v_i}$.

If v_j is a neighbor of v_i, then
$|\{\alpha \mid j(\alpha) = j\}| \le \alpha_{v_j v_i}$.
This implies that $d + 1 \le d_G^+(v_i)$.

Using the above conclusion, it is easy to see that *c* determines a circular $\Delta^+(G)$-coloring of *G*.

If the edge-weights are not integers, we proceed as follows. Let *N* be a large positive real number, and let $a'_{uv} = [Na_{uv}]$.

Denote by G' the weighted graph thus obtained. Clearly, $\chi_c(G') \ge N \cdot \chi_c(G)$.

By the above,

$$\chi_c(G') \le \Delta^+(G') = \max\left\{d_{G'}^+(v) + a'_{vu}\right\}$$

$\le N\Delta^+(G) + n$ Therefore, $\chi_c(G) \le \Delta^+(G) + \dfrac{n}{N}$. Since *N* is arbitrarily large, $\chi_c(G) \le \Delta^+(G)$.

Hence the proof.

II. CONCLUSION

We already know that the chromatic number of any planar graph, while it is known that the chromatic number of a planar graph is *NP*-complete, and also we present an infinite family of triangle – free planar graphs whose star chromaticnumber of color critical graphs.

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