Oscillatory Behavior Of Second Order Neutral Difference Equations

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Abstract- *In this research paper, we see a class of second order neutral delay difference equation of the form*

$$\Delta\left[r(n)\,|\Delta z(n)|^{\alpha-1}\,\Delta z(n)\right] + q(n)f(x(n-\sigma)) = 0; \quad n \ge n_0$$

Where $z(n) = x(n) - p(n)x(n-\tau),\ \alpha > 0$ *We determine sufficient conditions under which every solution of (*) is either oscillatory or tends to zero. Our consequences develop a number of correlated results reported in the literature.*

Keywords- Oscillation, non oscillation, asymptotic behavior, neutral, second order, difference equation

I. INTRODUCTION

The paper deals with the following second order nonlinear neutral difference equation of the form

$$\Delta\left[r(n)\,|\Delta z(n)|^{\alpha-1}\,\Delta z(n)\right] + q(n)f(x(n-\sigma)) = 0; \quad n \ge n_0$$

Where $z(n) = x(n) - p(n)x(n-\tau),\ \alpha > 0$ is a ratio of odd positive integers and Δ is the forward difference operator defined by $\Delta x(n) = x(n+1) - x(n).$ Throughout the paper, we assume the following condi-tions:

(H_1) $\{p(n)\}_{n=n_0}^{\infty}$ is a sequence of nonnegtive real numbers and there exists a constant p such that
$0 \le p(n) \le p < 1;$

(H_2) $\{q(n)\}$ is a sequence of nonnegative real numbers and $q(n)$ is not identically zero for large values of n;

(H_3) $\{r(n)\}$ is a sequence of positive real numbers;

(H_4) τ and σ are positive integers;

(H_5) $f : R \to R$ is a continuous function with the property that $uf(u) > 0$ for all $u \ne 0$ and there exists a constant $k > 0$ such that

$$\frac{f(u)}{|u|^{\alpha-1}u} \ge u; \quad for \quad u \ne 0.$$

Let $n^* =$ max $\{\tau, \sigma\}.$ For any real sequence $\{\theta(n)\}$ defined in $n_0 - n^* \le n \le n_0 - 1,$ the equation (1.1) has solution $\{x(n)\}$ defined for $n \ge n_0$ and satisfying the initial condition $x(n) = \theta(n)$ for $n_0 - n^* \le n \le n_0 - 1.$ A solution $\{x(n)\}$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative and non oscillatory otherwise.

Recently, there has been much interest in studying the oscillatory and asymptotic behavior of second order functional difference equations; see for example [3, 4, 6, 8, 9, 12–24]. For the general theory of difference equations, one can refer to [1, 2, 7]. Prior to presenting our oscillation and asymptotic criteria, we briefly comment results for (1.1) and its particular cases which motivated the present study.

Theorems on oscillatory and asymptotic behavior of second order nonlinear neutral difference equations

Saker et al. [14] investigated the oscillatory behavior of second order nonlinear difference equations of the form.

$$\Delta(r(n)(\Delta y(n))^{\alpha}) + p(n)\Delta(y(n))^{\alpha} + q(n)f(y(n+1)) = 0$$

and obtained sufficient conditions for oscillation of all solutions of (1.2).

Thandapani et al. [21] proved that every solution of the equation

$$\Delta^2(y(n-1) - py(n-1-k)) + q(n)f(y(n-l)) = 0$$

Is oscillatory if and only if

$$\sum_{n=1}^{\infty} f(n)q(n) = \infty.$$

and also established that every solution of (1.3) is oscillatory if

$$\liminf_{n\to\infty} \sum_{k=n-l}^{n-1} (k-l-1)q(s) > \frac{1}{M}\left(\frac{l}{l+1}\right)^{l+1}$$

Sternal et al. [15] established that every nonoscillatory solution of the equation

$$\Delta(r(n)\Delta(y(n)+p(n)y(n-\tau))+q(n)f(y(n-\sigma))$$

tends to zero as $n \to \infty$ under the conditions

$$\sum_{n=0}^{\infty} \frac{1}{r(n)} = \infty \quad and \quad \sum_{n=0}^{\infty} q(n) = \infty$$

Li et al. [11] investigated the second order neutral delay difference equation of the form

$$\Delta[q(n-1)\Delta(y(n-1)+p(n-1)y(n-1-\sigma)]$$
$$+q(n)f(y(n-\tau)) = 0$$

and derived sufficient conditions for oscillatory of all solutions of (1.8) under the condition

$$\Sigma \frac{1}{a(n)} = \infty.$$

Li et al. [12] consider the following second order nonlinear difference equation of the form

$$\Delta(r(n)(\Delta y(n))^{\alpha}) + p(n+1)f(y(n+1)) = 0$$

and established sufficient conditions for oscillation of every solution of (1.9).

In [11], we studied a second order nonlinear neutral delay difference equation of the form

$$\Delta[r(n)\Delta(y(n)-p(n)y(n-\tau))]$$
$$+q(n)f(y(n-\sigma)) = 0;$$

under the assumptions $0 \le p(n) \le p < 1$ and $\frac{f(u)}{u} \ge k > 0,$ for all $u \neq 0,$

$$\sum_{n=n_0}^{\infty} \frac{1}{r(n)} = \infty$$

And

$$\sum_{n=n_0}^{\infty} \frac{1}{r(n)} < \infty.$$

We proved that every solution of (1.10) is either oscillatory or tends to zero if $\sigma > \tau,$ (1.11) holds and there exists a sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ positive real numbers such that

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n-1} \left[k\eta(s)Q(s) - \frac{(1+p)r(s-\sigma)((\Delta\eta(s)_+)^2}{4\eta(s)}\right] = \infty.$$

Also, we proved that every solution of (1.10) is either oscillatory or tends to zero under the conditions $\sigma > \tau,$ (1.12) and if there exists a positive real valued sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ such that (1.13) holds and

$$\limsup_{s=n_0} \sum_{s=n_0}^{n-1} \left[kQ(s)\beta(s+1) - \frac{1+p}{4r(s)\beta(s+1)}\right] = \infty;$$

Where

$$\beta(n) = \sum_{s=n}^{\infty} \frac{1}{r(s)}.$$

Li et al. [10] studied the oscillatory behavior of a class of second order nonlinear neutral delay differential equations of the form $(r(t)\left|z'(r)\right|^{\alpha-1} z'(t)) + q(t)f(x(\sigma(t))) = 0$ and established sufficient conditions under every solution of (1.15) is oscillatory.

In this paper, we derive sufficient conditions which ensures that every solution of (1.1) is either oscillatory or tends to zero under the condition (1.11). Our work is motivated by Li etal. [10] and our present results are discrete analogous of will known results due to [10]. In the sequel, the following notation is frequently used:

$$Q(n) = \min\{q(n), q(n-\tau)\};$$
$$(u(n))_+ = \max\{0, u(n)\};$$

And

$$R(l,n) = \left(\sum_{s=l}^{n-\tau-\sigma-1} \frac{1}{(r(s))^{\frac{1}{\alpha}}}\right)\left(\sum_{s=l}^{n-1} \frac{1}{(r(s))^{\frac{1}{\alpha}}}\right)^{-1}.$$

II. SOME USEFUL LEMMAS

Lemma 2.1. [11]. Let $\{x(n)\}$ be an eventually positive solution of (1.1) and $\{z(n)\}$ be its associated sequence defined by

$$z(n) = x(n) - p(n)x(n-\tau).$$

If $\{\Delta z(n)\}$ is eventually negative or limsup $_{n\to\infty} x(n) > 0,$ then $z(n) > 0,$ eventually.

Theorems on oscillatory and asymptotic behavior of second order nonlinear neutral difference equations

Lemma 2.2.Assume that (1.3) holds, $let\ \{x(n)\}$ be an eventually positive solution of (1.1) such that lim sup $_{n\to\infty} x(n) > 0.$ Then its associated sequence $\{z(n)\}$ defined by (2.1) satisfies $\Delta z(n) > 0,$ eventually.

Proof. Assume that $\{x(n)\}$ be an eventually positive solution of (1.1) such that lims up $_{n\to\infty} x(n) > 0.$

Then by Lemma 2.1, we have $z(n) > 0.$ Also, from (1.1),

$$\Delta\left[r(n)\,|\Delta z(n)|^{\alpha-1}\,\Delta z(n)\right]$$
$$= -q(n)f(x(n-\sigma)$$
$$\leq -kq(n)x^{\alpha}(n-\sigma) \leq 0.$$

This shows that $\left\{r(n)\,|\Delta z(n)|^{\alpha-1}\,\Delta z(n)\right\}$ is eventually decreasing sequence. Consequently, we have $\Delta z(n) > 0$ or $\Delta z(n) < 0.$

If we let $\Delta z(n) < 0,$ then

$$r(n)\,|\Delta z(n)|^{\alpha-1}\,\Delta z(n) = r(n)\,(\Delta z(n))^{\alpha} \leq -c < 0.$$

Also, we have

$$\begin{aligned} z(n) - z(n_1) &= \sum_{s=n_1}^{n-1} \Delta z(s) \\ &= \sum_{s=n_1}^{n-1} \frac{(r(s)(\Delta z(s)^{\alpha})^{\frac{1}{\alpha}}}{(r(s))^{\frac{1}{\alpha}}} \\ &\leq (r(n_1)\,(\Delta z(n_1))^{\alpha})^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} \frac{1}{(r(s))^{\frac{1}{\alpha}}} \\ &< (-c)^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} \frac{1}{(r(s))^{\frac{1}{\alpha}}} \end{aligned}$$

Or

$$z(n) \leq z(n_1) + (-c)^{\frac{1}{\alpha}} \sum_{s=n_1}^{n-1} \frac{1}{(r(s))^{\frac{1}{\alpha}}},$$

which implies that $z(n) \to -\infty$ This is a contradiction to the fact that $z(n) > 0,$ eventually and the proof is complete.

Lemma 2.3. [7] If x and y are positive real numbers and $\lambda > 0,$ then

$$A^{\lambda} - B^{\lambda} \geq \lambda B^{\lambda-1}(A-B) \quad if \quad \lambda \geq 1$$

Or

$$A^{\lambda} - B^{\lambda} \geq \lambda A^{\lambda-1}(A-B) \quad if \quad 0 < \lambda \leq 1.$$

There is obviously equality when $\lambda = 1$ or $A = B.$

III. RESULTS

In this section we derive sufficient conditions under which every solution of (1.1) is either oscillatory or tends to zero.

Theorem 3.1. Assume that (1.3) holds. Suppose that there exists a sequence $\{\eta(n)\}_{n=n_0}^{\infty}$ of positive real numbers such that

$$\sum_{s=n_{**}}^{\infty} \left[\eta(s)Q(s)R^{\alpha}(n_*,s) - \frac{((\Delta\eta(s))_+)^{\alpha+1}}{2k(\alpha+1)^{\alpha+1}\eta^{\alpha}(s)}(r(s) + r(s-\tau))\right] = \infty,$$

for all sufficiently large n_* and for some $n_{**} \geq n_* \geq$ no then every solution of (1.1) is either oscillatory or tends to zero.

IV. PROOF

Assume the contrary. Without loss of generality, we may suppose that $\{x(n)\}$ is an eventually positive solution of (1.1) such that limsup $_{n\to\infty} x(n) > 0.$ Then by lemma 2.1, $z(n) > 0$ eventually where $z(n)$ is defined by (2.1). Then there exists an integer $n_1 \geq n_0$ such that for all $n \geq n_1,$ $x(n) > 0, x(n-\tau) > 0, x(n-\sigma) > 0$ and $z(n) > 0.$

Now, by Lemma 2.2 there exists an integer $n_2 \geq n_1$ such that $\Delta z(n) > 0$ fpr all $n \geq n_2.$ It follows from (1.1) that $\Delta(r(n)(\Delta z(n)^{\alpha}) \leq -kq(n)x^{\alpha}(n-\sigma) \leq 0,$ for all $n \geq n_1$

$$p_0^{\alpha}\Delta(r(n-\tau)(\Delta z(n-\tau))^{\alpha})$$
$$\leq -kq(n-\tau)z^{\alpha}(n-\tau-\sigma).$$

Combining the inequalities (3.4) and (3.5), we get

$$\Delta(r(n)(\Delta z(n))^{\alpha})+\Delta(r(n-\tau)(\Delta z(n-\tau))^{\alpha})$$
$$\leq -k(q(n)z^{\alpha}(n-\sigma)+p_0^{\alpha}q(n-\tau)z^{\alpha}(n-\tau-\sigma)$$
$$\leq -kQ(n)(z^{\alpha}(n-\sigma)+z^{\alpha}(n-\tau-\sigma)$$
$$\leq -2kQ(n)z^{\alpha}(n-\tau-\sigma)$$ f or all $n \geq n_3$.

Define a sequence $\{w(n)\}$

$$w(n)=\eta(n)\frac{r(n)(\Delta z(n))^{\alpha}}{z^{\alpha}(n)}.$$
$$-\eta(n)\frac{r(n+1)(\Delta z(n+1))^{\alpha}}{z^{\alpha}(n)z^{\alpha}(n+1)}\Delta z^{\alpha}(n)$$
$$+\frac{r(n+1)(\Delta z(n+1))^{\alpha}}{z^{\alpha}(n+1)}\Delta\eta(n).$$

We obtain contradiction with condition. This completes the proof.

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