Oscillatory Behavior Of Second-Order Nonlinear Neutral Differential Equations

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Abstract- *We study oscillatory behavior of solutions to a class of second-order nonlinear neutral differential equations under the assumptions that allow applications to differential equations with delayed and advanced arguments. New theorems do not need several restrictive assumptions required in related results reported in the literature. Several examples are provided to show that the results obtained are sharp even for second-order ordinary differential equations and improve related contributions to the subject.*

I. INTRODUCTION

This paper is concerned with the oscillation of a class of second-order nonlinear neutral functional differential equation

$$\left(r(t)\left((x(t)+p(t)x(\eta(t)))'\right)^{\gamma}\right)'+f(t,x(g(t)))=0, \quad (1)$$

Where $t \geq t_0 > 0$. The increasing interest in problems of the existence of oscillatory solutions to second-order neutral differential equations is motivated by their applications in theengineering and natural sciences. We refer the reader to [1–21] and the references cited therein.

We assume that the following hypotheses are satisfied:

(h_1) γ is a quotient of odd natural numbers, the functions $r, p \in C([t_0,\infty),\mathbb{R})$, and $r(t)>0$;

(h_2) the functions $\eta, g \in C([t_0,\infty),\mathbb{R})$ and $r(t)>0$;

$$\lim_{t\to\infty}\eta(t)=\lim_{t\to\infty}g(t)=\infty; \quad (2)$$

(h_3) the function $f(t,u)\in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$ satisfies $uf(t,u)>0$ (3)

for all $u\neq 0$ and there exists a positive continuous function $q(t)$ defined on $[t_0,\infty)$ such that

$$|f(t,u)|\geq q(t)|u|^{\gamma}. \quad (4)$$

by a solution of (1) we mean the function x defined on $[T_x,\infty)$ for some $T_x \geq t_0$ such that $x+p\cdot x\circ\eta$ and $r((x+p\cdot x\circ\eta)')^{\gamma}$ are continuously differentiable and x satisfies (1) for all $t \geq T_x$. In what follows, we assume that solutions of (1) exist and can be continued indefinitely to the right. Recall that a nontrivial solution x of (1)issaidtobeoscillatoryifitisnotofthesamesigneventually;other wise, it is called no oscillatory. Equation (1)is termed oscillatory if all its nontrivial solutions are oscillatory.

Recently, Baculikova and Dzurina [6] Studied oscillation of a second order natural functional
Differential equation

$$\left(r(t)\left(x(t)+p(t)x(\eta(t))\right)'\right)'+q(t)x(g(t))=0 \quad (5)$$

Assuming that the following (5) conditions hold

(H_1) $r,p,q\in C([t_0,\infty),\mathbb{R})$, $r(t)>0$, $0\leq p(t)\leq p_0<\infty$, and $q(t)>0$;

(H_2) $g\in C^1([t_0,\infty),\mathbb{R})$ and $\lim_{t\to\infty}g(t)=\infty$;

(H_3) $\eta\in C^1([t_0,\infty),\mathbb{R})$, $\eta'(t)\geq\eta_0>0$, and $\eta\circ g=g\circ\eta$.

They established oscillation criteria for (5) through the comparison with associated first-order delay differential inequalities in the case where

$$\int_{t_0}^{\infty} r^{-1}(t)\,dt=\infty. \quad (6)$$

Assuming that

$$\int_{t_0}^{\infty} r^{-1}(t)\,dt<\infty, \quad (7)$$

Han et al. [9], Li et al. [15], and Sun et al. [20]obtained oscillation results for (5), one of which we present below for the convenience of the reader. We use the notation

$$Q(t) := \min\{q(t), q(\eta(t))\},$$

$$\rho'_+(t) := \max\{0, \rho'(t)\},$$

$$\varphi(t) := \int_t^\infty r^{-1}(s)\,ds. \tag{8}$$

Theorem 1 (cf. [theorem 3.1] and [20,Theorem2.2]). Assume that conditions (H_1)–(H_3) and hold. Suppose also that $g(t) \le \eta(t) \le t$ and $g'(t) > 0$ for all $t \ge t_0$. If there exists a function $\rho \in C^1([t_0,\infty),(0,\infty))$ such that

$$\limsup_{t\to\infty} \int_{t_0}^t \left[\rho(s)Q(s) - \left(1+\frac{p_0}{\eta_0}\right)\frac{r(g(s))(\rho'_+(s))^2}{4\rho(s)g'(s)}\right] ds = \infty, \tag{9}$$

$$\limsup_{t\to\infty} \int_{t_0}^t \left[\varphi(s)Q(s) - \frac{1+(p_0/\eta_0)}{4\varphi(s)r(s)}\right] ds = \infty,$$

then (5) is oscillatory.

Replacing (6) with the condition

$$\int_{t_0}^\infty r^{-1/\gamma}(t)\,dt = \infty, \tag{10}$$

Baculikova and dzurina [7] extended results of [6]to a nonlinear neutral differential equation

$$\left(r(t)\left((x(t)+p(t)x(\tau(t)))'\right)^\gamma\right)' + q(t)x^\beta(\sigma(t)) = 0, \tag{11}$$

where β and γ are quotients of odd natural numbers. Hasanbulli and Rogovchenko [10] studied a more general second-order nonlinear neutral delay differential equation

$$\left(r(t)\left(x(t)+p(t)x(t-\tau)\right)'\right)' + q(t)f(x(t), x(\sigma(t))) = 0 \tag{12}$$

assuming that $0 \le p(t) \le 1, \sigma(t) \le t, \sigma'(t) > 0,$ and(6)holds. To introduce oscillation results obtained for (1)by Ere et al. [8], we need the following notation:

$$\mathbb{D} := \{(t,s) : t \ge s \ge t_0\},$$

$$\mathbb{D}_0 := \{(t,s) : t > s \ge t_0\},$$

$$h_-(t,s) := \max\{0, -h(t,s)\},$$

$$\theta(t,u) := \frac{\int_u^{g(t)} r^{-1/\gamma}(s)\,ds}{\int_u^t r^{-1/\gamma}(s)\,ds}. \tag{13}$$

We s ay t hat a continuous function $H : \mathbb{D} \to [0,\infty)$ belongs (i) $H(t,t) = 0$ for $t \ge t_0$ $H(t,s) > 0$

For $(t,s) \in \mathbb{D}_0$;

has a no positive continuous partial derivative $\partial H/\partial s$ with respect to the second variable satisfying

$$-\frac{\partial}{\partial s}H(t,s) - H(t,s)\frac{\delta'(s)}{\delta(s)} = \frac{h(t,s)}{\delta(s)}(H(t,s))^{\gamma/(\gamma+1)} \tag{14}$$

For some $h \in L_{\text{loc}}(\mathbb{D},\mathbb{R})$ and for some $\delta \in C^1([t_0,\infty), (0,\infty))$.

Theorem 2 (see [8,Theorem2.2,when $\mathbb{T} = \mathbb{R}$]). Let condition (10) and (h_1)–(h_3) hold. Suppose that $0 \le p(t) < 1, \eta(t) \le t,$ and $g(t) \ge t$ for all $t \ge t_0$. If there exists a function $H \in \mathfrak{H}$ such that, for all sufficiently large $T \ge t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \times \int_T^t \left[\delta(s)q(s)H(t,s)(1-p(g(s)))^\gamma \right. \tag{15}$$

$$-\frac{r(s)(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)}\Bigg]\, ds=\infty,$$

then (1) is oscillatory.

Theorem 3 (see [8,theorem 2,2 case $\mathbb{T}=\mathbb{R}$]). Let conditions (10) and (h_1)–(h_3) be satisfied also that $0 \le p(t) < 1$, $\eta(t) \le t$, and $g(t) \le t$ for all $t \ge t_0$. if there excite functions $H \in \mathfrak{H}$,such that for all sufficiently large $T_* \ge t_0$ and for some $T > T_*$,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)} \times \int_T^t \Bigg[\delta(s)\theta^\gamma(s,T_*)H(t,s)q(s)(1-p(g(s)))^\gamma \tag{16}$$

$$-\frac{r(s)(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^\gamma(s)}\Bigg]\, ds=\infty,$$

then (1) is oscillatory.
Assuming that

$$\int_{t_0}^\infty r^{-1/\gamma}(t)\,dt<\infty, \tag{17}$$

Li et al. [16] extended results of [10] to a nonlinear neutral delay differential equation

$$\left(r(t)\left((x(t)+p(t)x(t-\tau))'\right)^\gamma\right)' + q(t)f(x(t),x(\sigma(t)))=0, \tag{18}$$

Where $\gamma \ge 1$ γ=1is a ratio of odd natural numbers. Han et al. [9, Theorems 2.1 and 2.2] established sufficient conditions for

The oscillation of (1) provided that (17) is satisfied $0 \le p(t) <$ 1 and

$$\eta(t)=t-\tau\le t,\quad p'(t)\ge 0,\ g(t)\le t-\tau. \tag{19}$$

Xu and Mange [21] studied (1) under the assumptions that (17) holds $0 \le p(t) < 1$, and

$$\eta(t)=t-\tau\le t,\ p'(t)\ge 0, \tag{20}$$

obtaining sufficient conditions for all solutions of (1)either to be oscillatory or to satisfy $\lim_{t\to\infty}x(t)=0$; [21,Theorem 2.3]. Sacker [17] investigated oscillatory nature of (1) assuming that (17) is satisfied,

$$0\le p(t)<1,\ p'(t)\ge 0,\ \ g(t)\le\eta(t)\le t,\ \eta'(t)\ge 0, \tag{21}$$

And

$$\int_T^\infty\left(\frac{1}{r(s)}\int_T^s q(u)(1-p(u))^\gamma\varphi^\gamma(u)\,du\right)^{1/\gamma} ds=\infty$$

For some $T \ge t_0$, where (22)

$\varphi(u) := \int_u^\infty r^{-1/\gamma}(t)dt.$. Lietal. [12] studied oscillation of (1) under the conditions that (17)holds, η and g are strictly increasing $p(t) > 1$, and either or

$$g(t)\ge\eta(t)\quad\text{or}\quad g(t)\le\eta(t). \tag{23}$$

Li et al. [13] investigated(1) in the case where (H_1)–(H_3) hold $\gamma \ge 1, \eta(t) \ge t,$ and $g(t) \ge t.$ .In particular, sufficient conditions for all solutions of (1) either to be oscillatory or to satisfy $\lim_{t\to\infty}x(t)=0$ were obtainde under the assumpation that (17) holds and $0 \le p(t) \le p_1 < 1$; see [13,Theorem 3.8]. Sun et al. [19] established several oscillation results for (1) assuming that $(h_3), (H_1)$–$(H_3), (17),$ (17), and (23) are satisfied. The following notation is used in the next theorem:

$$Q(t):=\min\{q(t),q(\eta(t))\},$$
$$\rho'_+(t):=\max\{0,\rho'(t)\},$$

$$\varphi(t) := \int_{\tau(t)}^{\infty} r^{-1/\gamma}(s)\,ds.$$
(24)

Theorem 4 (see [19,Theorem4.1]). .Let conditions (h_3), (H_1)–(H_3), , and(17) be satisfied. Assume also that $\gamma \geq 1$, $g(t) \leq \eta(t) \leq t$, and *and* $g'(t) > 0$ for all $t \geq t_0$. suppose further that there exist functions $\rho \in C^1([t_0,\infty),(0,\infty))$ and $\tau \in C^1([t_0,\infty),\mathbb{R})$ such that $\tau(t) \geq t, \tau'(t) > 0$,

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{\rho(s)Q(s)}{2^{\gamma-1}} - \left(1+\frac{p_0{}^{\gamma}}{\eta_0}\right) \times \frac{r(g(s))\left(\rho'_+(s)\right)^{\gamma+1}}{(\gamma+1)^{\gamma+1}(\rho(s)g'(s))^{\gamma}} \right] ds = \infty,$$
(25)

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{\varphi^{\gamma}(s)Q(s)}{2^{\gamma-1}} - \left(1+\frac{p_0{}^{\gamma}}{\eta_0}\right) \times \frac{\gamma^{\gamma+1}\tau'(s)}{(\gamma+1)^{\gamma+1}\varphi(s)r^{1/\gamma}(\tau(s))} \right] ds = \infty.$$

Then (1) is oscillatory.

Our principal goal in this paper is to analyze the oscillatory behavior of solutions to (1)in the case where(17)holds and without assumptions

2. Oscillation Criteria

In what follows, all functional inequalities are tacitly assumed to hold for all t large enough, unless mentioned otherwise. We use the notation

$$z(t) := x(t) + p(t)x(\eta(t)),$$
$$R(t) := \int_{t}^{\infty} r^{-1/\gamma}(s)\,ds.$$
(26)

A continuous function $K : \mathbb{D} \to [0,\infty)$ is said to belong to the class $\mathfrak{K}$ if

(i) $K(t,t) = 0$ for $t \geq t_0$ and $K(t,s) > 0$ for $(t,s) \in \mathbb{D}_0$;
(Ii) k has a non positive continuous partial derivative $\partial K/\partial s$ with respect to the second variable satisfying

$$\frac{\partial}{\partial s}K(t,s) = -\zeta(t,s)(K(t,s))^{\gamma/(\gamma+1)}$$ for some $\zeta \in L_{\mathrm{loc}}(\mathbb{D},\mathbb{R})$.
(27)

Theorem 5. Let all assumptions of Theorem 2 be satisfied with condition (10) replaced by (17). Suppose that there exists a functions $\in C^1([t_0,\infty),(0,\infty))$ such that

$$\frac{m(t)}{r^{1/\gamma}(t)R(t)} + m'(t) \leq 0, \quad 1 - p(t)\frac{m(\eta(t))}{m(t)} > 0.$$
(28)

f there exists a function $K \in \mathfrak{K}$ such that, for all sufficiently large $T \geq t_0$,

$$\limsup_{t\to\infty} \int_{T}^{t} \left[K(t,s)q(s)\left(1 - p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma} \times \left(\frac{m(g(s))}{m(s)}\right)^{\gamma} - \frac{r(s)(\zeta(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}} \right] ds > 0,$$
(29)

then (1) is oscillatory.

Proof .Let $x(t)$ be no oscillatory solution of (1) without loss of generality we may assume there exits a $t_1 \geq t_0$
Such that $x(t) > 0, x(\eta(t)) > 0,$ and $x(g(t)) > 0$ for all $t \geq t_1$. Then $z(t) \geq x(t) > 0$ for all $t \geq t_1$, and by virtue of

$$\left(r(t)\left(z'(t)\right)^{\gamma}\right)' \leq -q(t)x^{\gamma}(g(t)) < 0,$$
(30)

The functions $r(t)(z'(t))^{\gamma}$ is strictly decreasing for all $t \geq t_1$. Hence $z'(t)$ does not change sign eventually; that is, there exists $t_2 \geq t_1$ such that either $z'(t) > 0$ or $z'(t) < 0$ for all $t \geq t_2$. We consider each of the two cases separately.

Case 1. Assume first that $z'(t) > 0$ for all $t \geq t_2$. Proceeding as in the proof of [8,Theorem 2.2,case $\mathbb{T} = \mathbb{R}$], we obtain a contradiction to (15).

Case 2. Assume now that $z'(t) < 0$ for all $t \geq t_2$. for $t \geq t_2$, define a new function $\omega(t)$ by

$$\omega(t) := \frac{r(t)\left(z'(t)\right)^{\gamma}}{z^{\gamma}(t)}.$$

(31)

Then $\omega(t) < 0$ for all $t \geq t_2$, and it follows from (30) that

$$z'(s) \leq \left(\frac{r(t)}{r(s)}\right)^{1/\gamma} z'(t)$$

(32)

For all $s \geq t \geq t_2$. Integrating (32) from t to $l, l \geq t \geq t_2$,

we

$$z(l) \leq z(t) + r^{1/\gamma}(t)\, z'(t) \int_t^l \frac{ds}{r^{1/\gamma}(s)}.$$

(33)

Passing to the limit as $l \to \infty$, we conclude that

$$z(t) + r^{1/\gamma}(t)\, z'(t)\, R(t) \geq 0,$$

(34)

which implies that

$$\frac{z'(t)}{z(t)} \geq -\frac{1}{r^{1/\gamma}(t)\, R(t)}.$$

(35)

Thus,

$$\left(\frac{z(t)}{m(t)}\right)' = \frac{z'(t)\, m(t) - z(t)\, m'(t)}{m^2(t)}$$

(36)

$$\geq -\frac{z(t)}{m^2(t)}\left[\frac{m(t)}{r^{1/\gamma}(t)\, R(t)} + m'(t)\right] \geq 0.$$

Consequently, there exists a $t_3 \geq t_2$ such that, for all $t \geq t_3$,

$$x(t) = z(t) - p(t)\, x(\eta(t)) \geq z(t) - p(t)\, z(\eta(t))$$

$$\geq z(t) - p(t)\frac{m(\eta(t))}{m(t)} z(t)$$

$$= \left(1 - p(t)\frac{m(\eta(t))}{m(t)}\right) z(t),$$

$$\frac{z(g(t))}{z(t)} \geq \frac{m(g(t))}{m(t)}.$$

Differentiating (31)and using(30), we have, for all $t \geq t_3$,

$$\omega'(t) \leq -q(t)\left(1 - p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right)^{\gamma}\left(\frac{m(g(t))}{m(t)}\right)^{\gamma}$$

(37)

$$-\frac{r(t)\left(z'(t)\right)^{\gamma}\left(z^{\gamma}(t)\right)'}{z^{2\gamma}(t)}$$

$$= -q(t)\left(1 - p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right)^{\gamma}\left(\frac{m(g(t))}{m(t)}\right)^{\gamma}$$

$$-\gamma\frac{r(t)\left(z'(t)\right)^{\gamma+1}}{z^{\gamma+1}(t)}.$$

(38)

Hence, by (31)and(38), we conclude that

$$\omega'(t) + q(t)\left(1 - p(g(t))\frac{m(\eta(g(t)))}{m(g(t))}\right)^{\gamma}\left(\frac{m(g(t))}{m(t)}\right)^{\gamma}$$

$$+ \gamma r^{-1/\gamma}(t)\, \omega^{(\gamma+1)/\gamma}(t) \leq 0$$

(39)

For all $t \geq t_3$. Multiplying (39) By $K(t,s)$ and integrating the resulting inequality from t_3 to t,
We obtain

$$\int_{t_3}^{t} K(t,s)\, q(s)\left(1 - p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma}$$

$$\times\left(\frac{m(g(s))}{m(s)}\right)^{\gamma} ds$$

$$\leq K(t,t_3)\,\omega(t_3) + \int_{t_3}^{t}\frac{\partial K(t,s)}{\partial s}\omega(s)\, ds$$

(40)

$$-\int_{t_3}^{t} \gamma K(t,s)\, r^{-1/\gamma}(s)\, \omega^{(\gamma+1)/\gamma}(s)\, ds$$

$$= K(t,t_3)\,\omega(t_3) - \int_{t_3}^{t}\zeta(t,s)\left(K(t,s)\right)^{\gamma/(\gamma+1)}\omega(s)\, ds$$

$$-\int_{t_3}^{t} \gamma K(t,s) r^{-1/\gamma}(s)(-\omega(s))^{(\gamma+1)/\gamma} ds.$$

In order to use the inequality

$$\frac{\gamma+1}{\gamma} AB^{1/\gamma} - A^{(\gamma+1)/\gamma} \le \frac{1}{\gamma} B^{(\gamma+1)/\gamma}, \quad \gamma > 0,\ A \ge 0,\ B \ge 0, \tag{41}$$

see Li et al. [16, Lemma 1 (ii)] for details; we let

$$A^{(\gamma+1)/\gamma} := \gamma K(t,s) r^{-1/\gamma}(s)(-\omega(s))^{(\gamma+1)/\gamma}, \tag{42}$$

$$B^{1/\gamma} := \frac{\gamma \zeta(t,s) r^{1/(\gamma+1)}(s)}{(\gamma+1)\gamma^{\gamma/(\gamma+1)}}.$$

Then, by virtue of (40), we conclude that

$$\int_{t_3}^{t} \left[K(t,s) q(s) \left(1 - p(g(s)) \frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma} \times \left(\frac{m(g(s))}{m(s)}\right)^{\gamma} - \frac{r(s)(\zeta(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}} \right] ds \tag{43}$$

$$\le K(t,t_3)\,\omega(t_3),$$

which contradicts (29). This completes the proof.

Theorem 6. Let all assumptions of Theorem 3 be satisfied with condition (10) replaced by (17). Suppose further that there exists a function $m \in C^1([t_0,\infty),(0,\infty))$ such that (28) holds. If there exists a function $K \in \mathfrak{K}$ such that, for all sufficiently large $T \ge t_0$,

$$\limsup_{t\to\infty} \int_{T}^{t} \left[K(t,s) q(s) \left(1 - p(g(s)) \frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma} - \frac{r(s)(\zeta(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}} \right] ds > 0, \tag{44}$$

then (1) is oscillatory.

Proof. The proof is similar to that of Theorem 5 and hence is omitted.

Theorem 7. Let conditions (10) and (h_1)–(h_3) be satisfied, $0 \le p(t) < 1$, $\eta(t) \ge t$, Assume that there exists a function $m \in C^1([t_0,\infty),(0,\infty))$ such that, for all sufficiently large $T_* \ge t_0$,

$$\frac{m(t)}{r^{1/\gamma}(t)\int_{T_*}^{t} r^{-1/\gamma}(s)\,ds} - m'(t) \le 0, \tag{45}$$

$$1 - p(t)\frac{m(\eta(t))}{m(t)} > 0.$$

If there exists a function $H \in \mathfrak{H}$ such that, for all sufficiently large $T \ge t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[\delta(s) q(s) H(t,s) \times \left(1 - p(g(s)) \frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma} \right. \tag{46}$$

$$\left. - \frac{r(s)(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)} \right] ds = \infty,$$

then (1) is oscillatory.

Proof. Without loss of generality, assume again that (1) possesses a nonoscillatory solution $x(t)$

Such that $x(t) > 0$, $x(\eta(t)) > 0$, and $x(g(t)) > 0$ on $[t_1,\infty)$ for some $t_1 \ge t_0$.

Then the $t \ge t_1$, (30) is satisfied and $z(t) \ge x(t) > 0$. it follows from (10) that there exists a

$T_* \ge t_1$ such that $z'(t) > 0$ for all $t \ge T_*$. By virtue of (30), we have

$$z(t) = z(T_*) + \int_{T_*}^{t} \frac{\left(r(s)(z'(s))^{\gamma}\right)^{1/\gamma}}{r^{1/\gamma}(s)} ds \tag{47}$$

$$\ge r^{1/\gamma}(t) z'(t) \int_{T_*}^{t} r^{-1/\gamma}(s)\,ds.$$

Since

$$\left(\frac{z(t)}{m(t)}\right)' = \frac{z'(t)\,m(t) - z(t)\,m'(t)}{m^2(t)}$$

$$\le \frac{z(t)}{m^2(t)}\left[\frac{m(t)}{r^{1/\gamma}(t)\int_{T_*}^{t} r^{-1/\gamma}(s)\,ds} - m'(t)\right] \le 0, \tag{48}$$

We conclude that

$$x(t) = z(t) - p(t)\,x(\eta(t))$$

$$\ge z(t) - p(t)\,z(\eta(t)) \tag{49}$$

$$\ge \left(1 - p(t)\frac{m(\eta(t))}{m(t)}\right) z(t).$$

For $t \ge T_*$, define a new function (t) by

$$u(t) := \delta(t)\,\frac{r(t)\left(z'(t)\right)^{\gamma}}{z^{\gamma}(t)}. \tag{50}$$

Then $u(t) > 0$ for all $t \ge T_*$, and the rest of the proof is similar to that of [8,Theorem2.2,case $\mathbb{T} = \mathbb{R}$]. This completes the proof.

Theorem 8. Let conditions (10) and (h_1)–(h_3) be satisfied. Suppose also that $0 \le p(t) < 1$, $\eta(t) \ge t$, $g(t) \le t$,

And there exists a function $m \in C^1([t_0,\infty),(0,\infty))$ such that (45) holds for all sufficiently large $T_* \ge t_0$. If there exists a function $H \in \mathfrak{H}$ such that, for some $T > T_*$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\delta(s)\,\theta^{\gamma}(s,T_*)\,q(s)\,H(t,s)\right.$$

$$\times \left(1 - p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma} \tag{51}$$

$$\left.-\frac{r(s)\,(h_-(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\delta^{\gamma}(s)}\right] ds = \infty,$$

then (1) is oscillatory.

Proof. The proof runs as in Theorem 7 and [8,Theorem2.2, case $\mathbb{T} = \mathbb{R}$] and thus is omitted

Theorem 9. Let all assumptions of Theorem 7 be satisfied with condition (10) replaced by (17). Suppose that there exist a function $K \in \mathfrak{K}$ and a function $\phi \in C^1([t_0,\infty),(0,\infty))$ such that

$$\frac{\phi(t)}{r^{1/\gamma}(t)\,R(t)} + \phi'(t) \le 0, \tag{52}$$

and, for all sufficiently large $T \ge t_0$,

$$\limsup_{t\to\infty} \int_T^t \left[K(t,s)\,q(s)\,(1 - p(g(s)))^{\gamma}\right.$$

$$\left.\times\left(\frac{\phi(g(s))}{\phi(s)}\right)^{\gamma} - \frac{r(s)\,(\zeta(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}}\right] ds > 0. \tag{53}$$

Then (1) is oscillatory.

Proof. Without loss of generality, assume as above that (1) possesses a nonoscillatory solution $x(t)$ such that $x(t) > 0$, $x(\eta(t)) > 0$, and $x(g(t)) > 0$ on $[t_1,\infty)$ for some $t_1 \ge t_0$. then for all $t \ge t_1$, (30) is satisfied and $z(t) \ge x(t) > 0$. Therefore, the function $r(t)(z'(t))^{\gamma}$ is strictly decreasing for all $t \ge t_1$, and so there exists $T_* \ge t_1$ such that either $z'(t) > 0$ or $z'(t) < 0$ for all $t \ge T_*$. As in the proof of Theorem 7, < we obtain a contradiction with (46). Assume now that $z'(t) < 0$ for all $t \ge T_*$. for $t \ge T_*$, define $\omega(t)$ By virtue of $\eta(t) \ge t$,

$$x(t) = z(t) - p(t)\,x(\eta(t))$$

$$\ge z(t) - p(t)\,z(\eta(t)) \tag{54}$$

$$\ge (1 - p(t))\,z(t).$$

The rest of the proof is similar to that of Theorem 5 and hence is omitted.

Theorem 10. Let all assumptions of Theorem 8 be satisfied with condition (10) replaced by (17). Suppose that there exists a function $K \in \mathfrak{K}$ such that, for all sufficiently large $T \ge t_0$,

$$\limsup_{t\to\infty} \int_T^t \left[K(t,s)\,q(s)\,(1 - p(g(s)))^{\gamma}\right. \tag{55}$$

$$-\frac{r(s)(\zeta(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}}\Bigg] \mathrm{d}s > 0.$$

Then (1) is oscillatory.

Proof. The proof resembles those of Theorems 5 and 9.

Remark 11. One can obtain from Theorems 5 and 6 various oscillation criteria by letting, for instance,

$$m(t) = R(t). \tag{56}$$

Likewise, several oscillation criteria are obtained from Theorems 7–10 with

$$m(t) = \int_{T_*}^{t} \frac{\mathrm{d}s}{r^{1/\gamma}(s)}, \qquad \phi(t) = R(t). \tag{57}$$

3. Examples and Discussion

The following examples illustrate applications of theoretical results presented in this paper.

Example 1. For $t \geq 1$, consider a neutral differential equation

$$\left(t^2\left(x(t) + p_0 x\left(\frac{t}{2}\right)\right)'\right)' + q_0 x(2t) = 0, \tag{58}$$

Where $p_0 \in (0, 1/2)$ and $q_0 > 0$ are constants. Here, $\gamma = 1$, $r(t) = t^2, p(t) = p_0, \eta(t) = t/2, q(t) = q_0$, and $g(t) = 2t$. Let $m(t) = t^{-1}$ and $K(t,s) = s^{-1}(t-s)^2$. then $\zeta(t,s) = 2s^{-1/2} + s^{-3/2}(t-s)$ and, for all sufficiently large $T \geq 1$ and for all q_0 satisfying $q_0(1-2p_0) > 1/2$, we have

$$\limsup_{t\to\infty} \int_T^t \left[K(t,s)\,q(s)\left(1 - p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma} \times \left(\frac{m(g(s))}{m(s)}\right)^{\gamma} - \frac{r(s)(\zeta(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}}\right] \mathrm{d}s$$

$$= \limsup_{t\to\infty} \int_T^t \left[\frac{q_0(1-2p_0)}{2}\frac{(t-s)^2}{s} - s - \frac{(t-s)^2}{4s} - (t-s)\right] \mathrm{d}s > 0. \tag{59}$$

On the other hand, letting $H(t,s) = s^{-1}(t-s)^2$ and $\delta(t) = 1$, we observe that condition (15) is satisfied for $q_0(1-2p_0) >$ 1/2.Hence,byTheorem 5,weconcludethat(58)is oscillatory provided that $q_0(1-2p_0) > 1/2.$ 1/2. Observe that results reported in [9, 12, 17, 21] cannot be applied to(58)since (58) since $p(t) < 1$ and conditions (19)–(22)fail to hold for this equation.

Example 2. For For $t \geq 1$, consider a neutral differential equation

$$\left(t^3\left(x(t) + \frac{1}{8}x\left(\frac{t}{2}\right)\right)'\right)' + q_0 t x\left(\frac{t}{3}\right) = 0, \tag{60}$$

Where $q_0 > 0$ is a constant, Here $\gamma = 1, r(t) = t^3, p(t) = 1/8, \eta(t) = t/2, q(t) = q_0 t,$ Let $m(t) = t^{-2}/2$ and $K(t,s) = s^{-2}(t-s)^2.$ then$\zeta(t,s) = 2s^{-1} + 2s^{-2}(t-s).$ Hence

$$\limsup_{t\to\infty} \int_T^t \left[K(t,s)\,q(s)\left(1 - p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma} - \frac{r(s)(\zeta(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}}\right] \mathrm{d}s$$

$$= \limsup_{t\to\infty} \int_T^t \left[\frac{q_0(t-s)^2}{2s} - s - \frac{(t-s)^2}{s} - 2(t-s)\right] \mathrm{d}s > 0 \tag{61}$$

Whenever $q_0 > 2.$ Let $H(t,s) = s^{-2}(t-s)^2$ and $\delta(t) = 1.$ .Then (16) is satisfied for $q_0 > 2.$

Therefore, using Theorem 6,w deduce that (60)is oscillatory if $q_0 > 2,$ so our oscillation result is sharper.

Example 3. For $t \geq 1$, consider the Euler differential equation

$$\left(t^2 x'(t)\right)' + q_0 x(t) = 0, \tag{62}$$

Where $q_0 > 0$ is a constant here $\gamma = 1, r(t) = t^2, p(t) = 0,\ q(t) = q_0,$ and $g(t) = t.$ choose $m(t) = t^{-1}$

And $K(t,s) = s^{-1}(t-s)^2.$ then $\zeta(t,s) = 2s^{-1/2} + s^{-3/2}(t-s),$ and so

$$\limsup_{t\to\infty} \int_T^t \left[K(t,s)\,q(s)\left(1 - p(g(s))\frac{m(\eta(g(s)))}{m(g(s))}\right)^{\gamma}\right.$$

$$\times\left(\frac{m\left(g\left(s\right)\right)}{m\left(s\right)}\right)^{\gamma}-\frac{r\left(s\right)\left(\zeta\left(t,s\right)\right)^{\gamma+1}}{\left(\gamma+1\right)^{\gamma+1}}\right]\mathrm{d}s$$

$$=\limsup_{t\to\infty}\int_T^t\left[\frac{q_0(t-s)^2}{s}-s-\frac{(t-s)^2}{4s}-(t-s)\right]\mathrm{d}s>0$$

(63)

provided that $q_0 > 1/4.$ Let $H(t,s) = s^{-1}(t-s)^2$ and $\delta(t) = 1.$ Then (15)holds for $q_0 > 1/4.$ it follows from Theorem 5 that (62)is oscillatory for $q_0 > 1/4,$ and it is well known that thiscondition is the best possible for the given equation. However, results of Saker [17]do not allow us to arrive at this conclusion due to condition(22).

Remark 12. In this paper, using an integral averaging technique, we derive several oscillation criteria for the second order neutral equation (1)in both cases(10)and(17).Contrary to [9, 12, 15, 17, 19–21], we do not impose restrictive (H_3) and (19)–(23) in our oscillation results. This leads to a certain improvement compared to the results in the cited papers. However, to obtain new results in the case where (17) holds, we have to impose an additional assumption on the function p; that this $p(t) < m(t)/m(\eta(t)).$ The question regarding the study of oscillatory properties of (1)with other methods that do not require this assumption remain open at the moment.

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