Study of Multiple Boolean Algebras-II

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Abstract- *In this paper the concept of 'Multiple Boolean Algebras' is further studied and isomorphism between power set of Multiple Boolean Algebras and n^{th} power of basic MBA is proved with a counter example proving that an improvement in the definition of MBA is necessary.*

Keywords- Multiple Boolean Algebra, Fuzzy set, Boolean Algebra, Multi-Valued logic

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I. INTRODUCTION

As an outgrowth of introduction of Fuzzy Sets by L. A. Zadeh in 1965, and then the introduction of Multiple Boolean Algebra by Silvano Di Zenzo[1], I dealt with the notion of MBA as introduced by him and attempted to give proofs of existence theorems in more detail in the last paper 'Study of Multiple Boolean algebra'[5] which is followed by some structure determining theorems. It was proved in detail that for a given pair of integers $p \geq 2$ and $n \geq 1$, there exists a MBA of order p and cardinality p^n. This 'Power set of multiple Boolean algebra' was further illustrated with examples in the last paper.

This time a notation, standard like those of group, ring and Boolean algebra is set up in the beginning and then it is proved that cross product of two MBAs is a MBA. Then isomorphism between two MBAs is defined and it is proved that power set of a finite MBA of cardinality p^n and order p is isomorphic to $[I(p)]^n$.

II. PRELIMINARIES

Zenzo showed that the set of all fuzzy subsets of a set becomes a Multiple Boolean Algebra if the binary operations on it are defined in a suitable manner. Adding a trivial axiom in his definition, it becomes as follows:

Let p be any integer greater than 1. Multiple Boolean algebra of order p is a set E with p binary operations 0, 1, 2, 3, ...,p-1; p distinguished elements $e_0, e_1, e_2, \ldots e_n$ and a bijection $U : E \rightarrow E$ such that the following axioms are satisfied :

For every x, y, z ∈ E and for every m=0, 1, 2, . . . , p-1,

MBA 1 $x \underline{m} x = x$
MBA 2 $x \underline{m} y = y \underline{m} x$
MBA 3 $(x \underline{m} y) \underline{m} z = x \underline{m} (y \underline{m} z)$
MBA 4 $x \underline{m} e_m = x$
MBA 5 for each $\underline{m}$, there exists $a_m \in E$ such that $x \underline{m} a_m = a_m$
MBA 6 $(x \, \underline{m+1} \, y) \underline{m} z = (x \underline{m} z) \, \underline{m+1} \, (y \underline{m} z)$
MBA 7 $U(x \underline{m} y) = U(x) \, \underline{m+1} \, U(y)$
MBA 8 $U^p(x) = x$ i.e. $U(U(U(\ldots U(U(x)) = x$
MBA 9 $x \underline{m} U(x) \underline{m} U^2(x) \underline{m} \ldots U^{p-1}(x) = a_m$
MBA 10 $a_i \neq a_j$ if $i \neq j$

where U is bijection from E onto E. (generalization of complement)

Note 1. It is understood that p^{th} operation is again 0^{th} operation cyclically.

Note 2. Axiom MBA 10 assumes the uniqueness of absorbing elements, and is my addition. How and why it is necessary to add it is cleared in this paper. Thus, there would be 10 axioms in total in the definition of MBA, 2 more than those in Zenzo's definition. Of course some of them can be derived from others, but it is better pre-assume to make the ensuing things less complicated. For example, the axiom MBA 4 can be proved using first three. Even after, dropping out the 4^{th} axiom, here it is proved that a new condition is yet required to make the concept full-proof, and a new definition of MBA is presented.

Note 3. The MBAs that have as much number of operations as they have elements in it are called by Zenzo as the Basic Multiple Boolean Algebras (BMBA)

III. NOTATION

Let us denote MBA of order p by a 5-touple $\langle E, \, m, e_m, \, a_m, \, U; \;\; 0 \leq m \leq p-1 \rangle$

Theorem 3.1:

Let
$\langle E, \underline{m}, e_{\underline{m}}, a_{\underline{m}}, U;\ 0 \le m \le p - 1 \rangle$ and $\langle \overline{E}, \overline{m}, e_{\overline{m}}, a_{\overline{m}}, U;\ 0 \le m \le p - 1 \rangle$

be two MBAs of same order p. Then the set $E \times \overline{E}$, with p binary operations $O_m\ (0 \le m \le p-1)$ defined by $(x, \bar{x})\ O_m\ (y, \overline{y}) = (x\ \underline{m}\ y,\ \bar{x}\ \overline{m}\ \overline{y})\ \forall x, y \in E$; $\bar{x}, \bar{y} \in \bar{E}$;

and p distinguished elements $(e_{\underline{i}}, e_{\bar{i}})\ i = 0\ to\ p-1$, and the fundamental isomorphism

U : $E \times \overline{E} \to E \times \overline{E}$ defined by U(x, $\bar{x}$) = (U(x) , $\overline{U}(\bar{x})$) is a MBA of order p

Proof : The definition of O_m itself is sufficient to prove the first three axioms of MBA. To prove that $(e_{\underline{m}}, e_{\overline{m}})$ works as identity element for operation O_m , see that

$(x, \bar{x})\ O_m\ (e_{\underline{m}}, e_{\overline{m}}) = ((x\ \underline{m}\ e_{\underline{m}}), (\bar{x}\ \overline{m}\ e_{\overline{m}})) = (x, \bar{x})$ $\forall (x, \bar{x}) \in E \times \overline{E}$

for the element $(e_{\underline{m}}, e_{\overline{m}})$ in $E \times \bar{E}$.

As for absorbing element in $E \times \bar{E}$, we see that for all $(x, \bar{x})$ in it

$$(x, \bar{x})\ O_m\ (a_{\underline{m}}, a_{\overline{m}}) = ((x\ \underline{m}\ a_{\underline{m}}), (\bar{x}\ \overline{m}\ a_{\overline{m}}) = (a_{\underline{m}}, a_{\overline{m}})$$

showing $(a_{\underline{m}}, a_{\overline{m}})$ is absorbing element for operation O_m. Their distinctness follows from the distinctness follows from the distinctness of the absorbing elements in factor algebras. As for the distributivity –

$$(x, \bar{x})\ O_m\ [(y, \bar{y})\ O_{m+1}\ (z, \bar{z})] = (x, \bar{x})\ O_m\ [(y\ \underline{m+1}\ z), (\bar{y}\ \overline{m+1}\ \bar{z})]$$

$$= \left(x\ \underline{m}(y\ \underline{m+1}\ z), \bar{x}\ \overline{m}\ (\bar{y}\ \overline{m+1}\ \bar{z})\right)$$

$$= \left((x\ \underline{m}\ y)\underline{m+1}\ (x\ \underline{m}\ z), (\bar{x}\ \overline{m}\ \bar{y})\overline{m+1}(\bar{x}\ \overline{m}\ \bar{z})\right)$$

$$= \left((x\ \underline{m}\ y, \bar{x}\ \overline{m}\ \bar{y})\ O_{m+1}\ (x\ \underline{m}\ z, \bar{x}\ \overline{m}\ \bar{z})\right)$$

$$= \left((x, \bar{x})\ O_m\ (y, \bar{y})\right)\ O_{m+1}\ [\ (x, \bar{x})\ O_m\ (z, \bar{z})$$

where x,y,z Є E and $\bar{x}, \bar{y}, \bar{z} \in \bar{E}$

For De Morgan's laws, consider,

$u[(x, \bar{x})\ O_m\ (y, \bar{y})] = u[(x\ \underline{m}\ y, \bar{x}\ \overline{m}\ \bar{y})] = [(u(x\ \underline{m}\ y), u(\bar{x}\ \overline{m}\ \bar{y})]$

$= (ux\ \underline{m+1}\ uy,\ \overline{u\,x}\ \overline{m+1}\ \overline{uy})$ $= (ux, \overline{u\,x})\ O_{m+1}\ (uy, \overline{uy}) = u(x, \bar{x})\ O_{m+1}\ u(y, \bar{y})$

To prove MBA 8, consider,

$u^p(x, \bar{x}) = u^{p-1}(u(x, \bar{x}))$

$= u^{p-1}(u(x), \bar{u}(\bar{x}))$

$= u^{p-2}(u^2(x), \bar{u}^2(\bar{x}))$

$= \ldots$

$= u(u^{p-1}(x), \bar{u}^{p-1}(\bar{x}))$

$= (u^p(x), \bar{u}^p(\bar{x}))$

$= (x, \bar{x})$ for all $(x, \bar{x}) \in E \times \bar{E}$

Similarly, using properties in E and $\bar{E}$,

$(x, \bar{x})\ O_m\ u(x, \bar{x})\ O_m\ u^2(x, \bar{x})\ O_m\ u^3(x, \bar{x}) \ldots\ O_m\ u^{p-1}(x,$

can be proved.

Note 4: If E & $\bar{\text{E}}$ are finite multiple Boolean algebras of order p and cardinality m and n respectively, then, $\text{E} \times \bar{\text{E}}$ is MBA of order p and cardinality $m \times n$. Particularly, I(p) $\times$ I(p) is MBA of order p and cardinality p^2. By induction, the theorem can be extended for any finite number of MBAs. Thus,

$[I(p)]^n = I(p) \times I(p) \times I(p) \ldots I(p)$ (n times)
is a MBA of order p and cardinality p^n.

Definition 3.1 : Let $\langle E, \underline{m}, e_{\underline{m}}, a_{\underline{m}}, u_j;0 \le m \le p - 1 \rangle$ and $\langle \bar{E}, \overline{m}, e_{\overline{m}}, a_{\overline{m}}, v_j;0 \le m \le p - 1 \rangle$

be two MBAs of equal order p. We say that they are isomorphic if there is a bijection $\emptyset : E \to \bar{E}$ which preserves identities, each operation $\underline{m}$, and the fundamental isomorphism u. That is,

I1. $\emptyset\left(x\ \underline{m}\ y\right) = \emptyset(x)\ \underline{m}\ \emptyset(y)\ \forall x,y \in E$

I2. $\emptyset\left(e_{\underline{m}}\right) = e_{\overline{m}}\ (m = 0,1,2,3,\ldots,p-1)$

I3. $\emptyset\left(u(x)\right) = v(\emptyset(x))\ \forall x \in E$

Note 5: The absorbing elements $a_{\underline{m}}$ get preserved by the isomorphism $\emptyset$, for, by MBA 9,

$$\emptyset\left(a_{\underline{m}}\right) = \emptyset\left(x\ \underline{m}\ u(x)\ \underline{m}\ u^2(x)\ \underline{m}\ u^3(x)\ldots\ u^{p-1}(x)\right)\ \forall x \in E$$

$$= \emptyset(x)\ \overline{m}\ \emptyset\left(u(x)\right)\ \overline{m}\ \emptyset(u^2(x))\ \overline{m}\ldots\overline{m}\ \emptyset(u^{p-1}(x))$$ … by I1.

$$= \emptyset(x)\ \overline{m}\ v\left(\emptyset(x)\right)\ \overline{m}\ v^2(\emptyset(x))\ \overline{m}\ldots\overline{m}\ v^{p-1}(\emptyset(x))$$ … by I3.

$$= a_{\overline{m}}$$

Lemma 3.2: $\emptyset\left(u^{-1}(x)\right) = v^{-1}(\emptyset(x)),\ x \in E$

Proof : By I3,

$$\emptyset\left(u(x)\right) = v(\emptyset(x)) \Rightarrow v^{-1}\left(\emptyset\left(u(x)\right)\right) = \emptyset(x)\ \ldots (A)$$

$$Hence,\ v^{-1}(\emptyset(x)) = v^{-1}\left(\emptyset\left(u(u^{-1}(x))\right)\right)$$

$$= \emptyset(u^{-1}(x))$$ by A

Note 6: If the isomorphism preserves any one of the p operations, then, it preserves all other operations. For, suppose $\emptyset$ preserves an operation $\underline{m}$,

i.e. $\emptyset(x\ \underline{m}\ y) = \emptyset(x)\overline{m}\,\emptyset(y)\ \forall x,y \in E$

Then, $\emptyset(x\ \underline{m+1}\ y)$

$= \emptyset\left(u(u^{-1}(x)\ \underline{m}\ u^{-1}(y))\right)$ by De Morgan's Law

$= v\left[\emptyset(u^{-1}(x)\ \underline{m}\ u^{-1}(y))\right]$

$= v\left[\emptyset(u^{-1}x)\ \overline{m}\ (u^{-1}y)\right]$ by assumption

$= v\left[v^{-1}(\emptyset(x))\ \overline{m}\ v^{-1}(\emptyset(y))\right]$ by Lemma

$= v\left(v^{-1}(\emptyset(x))\right)\ \overline{m+1}\ v\left(v^{-1}(\emptyset(y))\right)$

$= \emptyset(x)\ \overline{m+1}\ \emptyset(y)\ \forall x,y \in E$

Observation : In ordinary Boolean algebras, we have the result that, the power set of finite Boolean algebra which is of cardinality 2^n is isomorphic to the Boolean algebra B_2^n where B_2 is the smallest Boolean algebra $\{0,1\}$. Here we prove that the power set MBA of order p and cardinality p^n is isomorphic to $[I(p)]^n$.

Theorem 3.2 : The power set MBA of order p and cardinality p^n denoted by

$\langle \overline{E}, \overline{m}, f_{\overline{m}}, L_{\overline{m}}, V \rangle$

and the MBA $[I(p)]^n$ (formed by cross product of I(p) n times) are isomorphic.

Proof : The MBA E is the set of all functions f: A→ I(p) where A is a finite set of cardinality n.

Let, A = { $x_1, x_2, \ldots, x_n$}

The identity element for operation $\overline{m}$ on E is the function –

$f_{\overline{m}} : A \to I(p)\ \ defined\ by\ f_{\overline{m}}(x) = e_m\ \forall x_i \in A.$

and absorbing element $L_{\overline{m}}$ is the function given by

$L_{\overline{m}}(x) = a_m\ \forall x_i \in A.\quad 0 \le m \le p-1$

where $e_0, e_1, \ldots e_{p-1}$ are identities in I(p) and $a_0, a_1, \ldots a_{p-1}$ are absorbing elements in I(p)

The fundamental isomorphism V in E is given by,

V(f) = g iff g(x) = u(f(x)) for all x in E . . . . (1)

Where u is fundamental isomorphism in I(p).

Let us remind that $\underline{0}, \underline{1}, \underline{2}, \ldots, \underline{p-1}$ are the p operations in I(p). Let $O_1, O_2, O_3 .. O_{p-1}$ be the operations in $[I(p)]^n$. Note that they are defined as:

$(b_1, b_2, b_3, \ldots, b_n)\ O_m\ (d_1, d_2, d_3, \ldots, d_n) = (b_1\ \underline{m}\ d_1, b_2\ \underline{m}\ d_2, \ldots b_n\ \underline{m}\ d_n)$ . . . (2)

where b_i , $d_i \in I(p)$. Also, the fundamental isomorphism U in $[I(p)]^n$ is given by:

$U[(b_1, b_2, \ldots, b_n)] = [u(b_1), u(b_2), \ldots, u(b_n)]\ \forall (b_1, b_2, \ldots, b_n) \in [I(p)]^n$ . . . . . (3)

Identity element for the operation O_m is the element $(e_m, e_m, \ldots, e_m)$ and the absorbing element for O_m is $(a_m, a_m, \ldots, a_m)$ in $[I(p)]^n$. With these pre-requisites, now define a function $\emptyset : E \to [I(p)]^n$ by

$$\emptyset(f) = (f(x_1), f(x_2), f(x_3), \ldots, f(x_n))\quad \forall f \in E$$

We prove that $\emptyset$ is isomorphism.

Claim 1: $\emptyset$ is well defined.

Proof: Let $\emptyset(f) \neq \emptyset(g), where\ f, g \in E$

$\Rightarrow$ $(f(x_1), f(x_2), f(x_3), \ldots, f(x_n) \neq (g(x_1), g(x_2), g(x_3), \ldots, g(x_n))$

for some f & g $\in E$

$\Rightarrow (f(x_i)) \neq (g(x_i))$ for some $0 \leq i \leq n$

$\Rightarrow$ f $\neq$ g

Thus, $\emptyset$ is well defined.

Claim 2: $\emptyset$ is one-one and onto i.e. bijection.

Proof: Let $\emptyset(f) = \emptyset(g)$

$\Rightarrow$ $(f(x_1), f(x_2), f(x_3), \ldots, f(x_n) = (g(x_1), g(x_2), g(x_3), \ldots, g(x_n))$

$\Rightarrow f(x_i) = g(x_i)\ where\ 0 \leq i \leq n$

$\Rightarrow$ f = g.

Hence $\emptyset$ is one-one.

Now for any (b_1, b_2, b_3, …, b_n) $\in [I(p)]^n$ the function $f : A \rightarrow [I(p)]$ defined by

$f(x_i) = b_i$ gives $\emptyset(f) =$ (b_1, b_2, b_3, …, b_n). Note that A = { x_1, x_2, … , x_n }

Thus, $\emptyset$ is one-one and onto i.e. bijection.

Claim 3: $\emptyset$ preserves each operation i.e. $\emptyset(f\ \overline{m}\ g) = \emptyset(f)\ O_m\ \emptyset(g),\ \forall f\ \&\ g \in E$

Proof: By definition,

$$\emptyset(f\ \overline{m}\ g) = ((f\ \overline{m}\ g)(x_1), (f\ \overline{m}\ g)(x_2), \ldots, (f\ \overline{m}\ g)(x_n))$$

$$= (f(x_1)\ \underline{m}\ g(x_1), f(x_2)\ \underline{m}\ g(x_2), \ldots, f(x_n)\ \underline{m}\ g(x_n))$$

$$= (f(x_1), f(x_2), \ldots, f(x_n))\ O_m\ (g(x_1), g(x_2), \ldots, g(x_n))$$ by (2)

$$= \emptyset(f)\ O_m\ \emptyset(g)$$

Thus, $\emptyset$ preserves each operation.

Claim 4: $\emptyset(v(f)) = U(\emptyset(f))\ \forall f \in E$

Proof : : $\emptyset(v(f))$

$$= ((v(f))(x_1), (v(f))(x_2), \ldots, (v(f))(x_n))$$

$$= (u(f(x_1)), u(f(x_2)), \ldots, u(f(x_n)))$$ by (1)

$$= U(f(x_1), f(x_2), \ldots, f(x_n))$$ by (3)

$$= U(\emptyset(f))$$

Claim 5: Identities are preserved by $\emptyset$.

i.e. to prove that $\emptyset(f_m) = (e_m, e_m, e_m, \ldots, e_m)$ for all m = 0, 1, 2, 3, …, p-1

Proof :

$$\emptyset(f_m) = (f_m(x_1), f_m(x_2), f_m(x_3), \ldots, f_m(x_n))$$

$$= (e_m, e_m, e_m, \ldots, e_m)$$ by definition of f_m

Here, $(e_m, e_m, e_m, \ldots, e_m)$ is identity for the operation O_m in $[I(p)]^n$.

Thus, above proved five claims collectively prove that $\emptyset$ is isomorphism.

Why MBA is a generalization of Boolean Algebra?

In Boolean algebra, only the equality of cardinality of two Boolean algebras is sufficient for making them isomorphic. Here we have proved that the power set finite MBA of cardinality p^n and order p is isomorphic with $[I(p)]^n$. But that does not imply that any two MBAs with equal cardinality and equal order are isomorphic! We are presenting a counter example of two non-isomorphic MBAs having order 4 and cardinality 4. Well, of course that imples either the definition of MBA has to be improved or keep the present MBA given by Zenzo Di Silvano as a generalized Boolean algebra with the ordinary Boolean algebra as a special case of it having order 2.

Example 3.1: Let F = { t_0, t_1, t_2, t_3 } be a set of cardinality 4 with 4 binary operations $\overline{0}, \overline{1}, \overline{2}, \overline{3}$ defined as follows –

$\overline{0}$	t_0	t_1	t_2	t_3
t_0	t_0	t_2	t_2	t_0
t_1	t_2	t_1	t_2	t_1
t_2	t_2	t_2	t_2	t_2
t_3	t_0	t_1	t_2	t_3

$\overline{1}$	t_0	t_1	t_2	t_3
t_0	t_0	t_1	t_2	t_3
t_1	t_1	t_1	t_1	t_1
t_2	t_2	t_1	t_2	t_1
t_3	t_3	t_1	t_1	t_3

$\overline{2}$	t_0	t_1	t_2	t_3
t_0	t_0	t_3	t_0	t_3
t_1	t_3	t_1	t_1	t_3
t_2	t_0	t_1	t_2	t_3
t_3	t_3	t_3	t_3	t_3

$\overline{3}$	t_0	t_1	t_2	t_3
t_0	t_0	t_0	t_0	t_0
t_1	t_0	t_1	t_2	t_3
t_2	t_0	t_2	t_2	t_0
t_3	t_0	t_3	t_0	t_3

Now, let v: F → F be the bijection defined by
$v(t_0) = t_2$, $v(t_1) = t_3$, $v(t_2) = t_1$, $v(t_3) = t_0$

Then, F is a multiple Boolean algebra of order 4 and cardinality 4 where the identity elements are
$e_{\overline{0}} = t_3$, $e_{\overline{1}} = t_0$, $e_{\overline{2}} = t_2$, $e_{\overline{3}} = t_1$
And absorbing elements are
$a_{\overline{0}} = t_2$, $a_{\overline{1}} = t_1$, $a_{\overline{2}} = t_3$, $a_{\overline{3}} = t_0$

One can easily verify that F is a Multiple Boolean Algebra. But this MBA is not isomorphic with the basic MBA I(4). Because any isomorphism $\emptyset$ between F and I(4) must preserve identities. This means one must define $\emptyset: I(4) \rightarrow F$ such that
$\emptyset(0) = t_0, \emptyset(1) = t_1$, $\emptyset(2) = t_2$, $and\ \emptyset(4) = t_4$
But then,
$\emptyset(1\ \underline{0}\ 0) = \emptyset(0) = t_0\ while\ \emptyset(1)\ \underline{0}\ \ \emptyset(0) = t_1\,\overline{0}\,t_0 = t_2$

Thus, $\emptyset$ does **not** preserve operations.
Hence F and I(4) are *not* isomorphic.

Observation : In the above define MBA named F, we have $e_{\underline{m+2}} = a_m\ \ for\ m = 0, 1, 2, 3$.

i.e. identity of $(m+2)^{th}$ operation = absorbing element of $\overline{m}$. The difficulty that the above example raised for us shows that, there is yet a room to improve the definition of Multiple Boolean Algebra. We see that in ordinary Boolean Algebra, the set of identities is equal to the set of absorbing elements; a fact which Zenzo has neither assumed nor it can be proved using his axioms. But this construction reveal that the set of identities and absorbing elements are equal. Moreover his examples exhibit that
identity of $(m+1)^{th}$ operation = absorbing element of m^{th} operation.
i.e. $e_{m+1} = a_m$
which is true in case of ordinary Boolean algebras.

Hence we suggest that the above condition be added to the definition of MBA given by Zenzo. Then the new definition of MBA will be as follows –

Definition 3.2: Let p be an integer greater than 1. A Multiple Boolean algebra of order p is a set E with p binary operations $\underline{0}, \underline{1}, \underline{2}, \ldots \underline{p-1}$ defined on it, and p distinguished elements $e_0, e_1, e_2, \ldots, e_{p-1}$ and a bijection u: E → E such that the following axioms are satisfied :

For every x,y,z in E, for every m = 0, 1, 2, . . ., p-1

A1 $x \,\underline{m}\, x = x$

A2 $x \,\underline{m}\, y = y \,\underline{m}\, x$

A3 $x \,\underline{m}\, (y \,\underline{m}\, z) = (x \,\underline{m}\, y) \,\underline{m}\, z$

A4 $x \,\underline{m}\, e_m = x$

A5 $x \,\underline{m}\, e_{m+1} = e_{m+1}$

A6 $x \,\underline{m}\, (y \,\underline{m+1}\, z) = (x \,\underline{m}\, y) \,\underline{m+1}\, (x \,\underline{m}\, z)$

A7 $u(x \,\underline{m}\, y) = u(x) \,\underline{m+1}\, u(y)$

A8 $u^p(x) = x$

A9 $x \,\underline{m}\, u(x) \,\underline{m}\, u^2(x) \,\underline{m} \ldots u^{p-1}(x) = e_{m+1}$

Then according to this definition, the algebra defined in example 3 will *no more* be a MBA.

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