Extension of Peano's Existence Theorem For A Second Order Initial Value Problem

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Abstract- *Several existence theorems are known for a first order Initial Value Problem (IVP). In this paper we extend the existence of Peano's theorem to a second order Initial Value Problem ([6], [7]) of the type* $y'' = f(x,y),\quad y(x_0) = y_0,\quad y'(x_0) = y_1,$ *by suitably constructing a sequence which is equicontinuous and uniformly bounded, so as to apply Ascoli-Arzela theorem.*

Keywords- Ascoli-Arzela, equicontinuous, Initial Value Problem, Integral equations, Lipschitz condition, sandwich lemma, sequence, subsequence, uniform bound.

I. INTRODUCTION

The existence theorem due to Peano ([3], [4]) for IVPs of the type $y' = f(x,y),\quad y(x_0) = y_0$ have been extended to IVPs of the type $y'' = f(x,y),\quad y(x_0) = y_0,\quad y'(x_0) = y_1,$ by suitably constructing a sequence which is equicontinuous and uniformly bounded, so as to apply Ascoli-Arzela theorem.

II. PRELIMINARY RESULTS

Consider the first order IVP,

$$y' = f(x,y),\quad y(x_0) = y_0$$
(1)

This first order IVP is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$
(2)

The Peano Existence theorem for first order IVPs is stated as

Theorem 1 ([1]) (Global Existence Theorem) Let $f(x,y)$ be continuous and bounded in the strip $T: |x - x_0| \le a,\quad |y| < \infty$. Then the IVP (1) has at least one solution in $|x - x_0| \le a$.

Theorem 2 ([2]) (Local Existence Theorem) Let $f(x,y)$ be continuous in $\overline{S}: |x - x_0| \le a,\quad |y - y_0| \le b,$ and hence $\exists\, M > 0,$ such that, $|f(x,y)| \le M, \forall\, (x,y) \in \overline{S}.$ Then the IVP (1), has at least one solution in $J_h: |x - x_0| \le h = \min\{a, \frac{b}{M}\}$

III. EXTENSION OF EXISTENCE THEOREM

Consider the second order IVP,

$$y'' = f(x,y),\quad y(x_0) = y_0,\quad y'(x_0) = y_1$$
(3)

The solution of (3) is equivalent to solving the integral equation,

$$y(x) = y_0 + (x - x_0)y_1 + \int_{x_0}^{x} (x - t) f(t, y(t))\, dt$$
(4)

Theorem 3 Let $f(x,y)$ be continuous and bounded in the strip $T: |x - x_0| \le a,\quad |y| < \infty.$ Then the IVP (2) has at least one solution in $|x - x_0| \le a.$

Proof We shall give an existence proof in $[x_0, x_0 + a]$ and its existence to $[x_0 - a, x_0]$ is immediate.

First, divide the interval $[x_0, x_0 + a]$ into $m + 1$ subintervals of equal length, by taking m points in a partition of $[x_0, x_0 + a]$.

Choose m such that $m > a.$
Define,

$$y_m(x) = \begin{cases} y_0 + (x - x_0)y_1, \quad x_0 \le x \le x_0 + \frac{a}{m} \\ y_0 + (x - x_0)y_1 + \int_{x_0}^{x} (x - t) f\big(t, y(t)\big) dt, \\ x_0 + \frac{ka}{m} \le x \le x_0 + \frac{(k+1)a}{m}, \; k = 1,2, \dots m - 1 \end{cases}$$

The length of each subinterval is now $\frac{a}{m}$ which is less than 1.

The sequence $y_m(x)$ defined in this manner is continuous because,

$$\lim_{x \to (x_0 + \frac{a}{m})^-} y_m(x) = y_0 + \frac{a y_1}{m} = \lim_{x \to (x_0 + \frac{a}{m})^+} y_m(x).$$

Further, $y_m(x_0) = y_0, \; y_m'(x_0) = y_1$.

Since f is bounded in strip T, $\exists\, M > 0$ such that $|f(x, y)| \le M \;\; \forall (X, Y) \in T$.

We claim that the sequence $\{y_m(x)\}$ is equicontinuous.
Let $x_1, x_2 \in [x_0, x_0 + a]$.

Let $K = |y_1| + M$.

We have the following 3 cases depending upon the subintervals in which x_1, x_2 lie.

Case 1: If $x_1, x_2 \in \left[x_0, x_0 + \frac{a}{m}\right]$,

$$\begin{aligned} |y_m(x_2) - y_m(x_1)| &= |(y_0 + (x_2 - x_0)y_1) - (y_0 + (x_1 - x_0)y_1)| \\ &= |y_1||x_2 - x_1| \\ &\le K\, |x_2 - x_1| \end{aligned}$$

Case 2: If $x_1 \in \left[x_0, x_0 + \frac{a}{m}\right], \; x_2 \in \left[x_0 + \frac{ka}{m}, x_0 + \frac{(k+1)a}{m}\right]$, $k = 1,2, \dots, m - 1$.

Then,

$$|y_m(x_2) - y_m(x_1)| = |(y_0 + (x_2 - x_0)y_1 + \int_{x_0}^{x_2 - \frac{a}{m}} (x_2 - t) f\big(t, y_m(t)\big) dt) - (y_0 + (x_1 - x_0)y_1)|$$

$$\le |y_1||x_2 - x_1| + M \int_{x_0}^{x_2 - \frac{a}{m}} |t - x_2| dt$$

$$\le |y_1||x_2 - x_1| + \frac{M}{2} |x_2 - x_1|^2$$

$$\le |y_1||x_2 - x_1| + \frac{M}{2} |x_2 - x_1|$$

$$\le K\, |x_2 - x_1|$$

Case3:

If $x_1, x_2 \in \left[x_0 + \frac{ka}{m}, x_0 + \frac{(k+1)a}{m}\right], \; k = 1,2, \dots, m - 1.$
Then,

$$\begin{aligned} | y_m(x_2) - y_m(x_1) | = | &(y_0 + (x_2 - x_0) y_1 \\ &+ \int_{x_0}^{x_2 - \frac{a}{m}} (x_2 - t) f(t, y_m(t)) dt) - \\ &(y_0 + (x_1 - x_0) y_1 \\ &+ \int_{x_0}^{x_1 - \frac{a}{m}} (x_1 - t) f(t, y_m(t)) dt) | \end{aligned}$$

$$\le |y_1||x_2 - x_1| + M \int_{x_0}^{x_2 - \frac{a}{m}} |t - x_2| dt + M \int_{x_1 - \frac{a}{m}}^{x_0} |t - x_1| dt$$

$$\le |y_1||x_2 - x_1| + M |x_2 - x_1|^2$$

$$\le |y_1||x_2 - x_1| + \frac{M}{2} |x_2 - x_1|$$

$$\le K |x_2 - x_1|$$

In all cases,
$|y_m(x_2) - y_m(x_1)| \le K |x_2 - x_1| \;\; \forall \; x \in [x_0, x_0 + a]$
. Therefore, the sequence $\{y_m(x)\}$ is equicontinuous.
Further,

$$|y_m(x)| = |y_0 + (x - x_0)y_1 + \int_{x_0}^{x - \frac{a}{m}} (x - t) f\big(t, y_m(t)\big) dt|$$

$$\le |y_0| + |y_1| a + M \int_{x_0}^{x - \frac{a}{m}} |t - x| dt$$

$$\le |y_0| + |y_1| a + \frac{M}{2} |x - x_0|^2$$

$$\le |y_0| + |y_1| a + \frac{M}{2} a^2$$

Therefore, the sequence $\{y_m(x)\}$ is uniformly bounded in $[x_0, x_0 + a]$. Being, equicontinuous and uniformly bounded in $[x_0, x_0 + a]$, by Ascoli-Arzela theorem, the sequence $\{y_m(x)\}$ contains a subsequence

$\{y_{m_p}(x)\}$ which converges uniformly in $[x_0, x_0 + a]$ to a continuous function $y(x)$.

We claim that, $y(x)$ is a solution of (3).

Now,

$$y_{m_p}(x) = y_0 + (x - x_0)y_1 + \int_{x_0}^{x - \frac{a}{m_p}} (x - t) f(t, y_m(t)) dt$$

$$= y_0 + (x - x_0)y_1 + \int_{x_0}^{x} (x - t) f\left(t, y_{m_p}(t)\right) dt - \int_{x - \frac{a}{m_p}}^{x} (x - t) f\left(t, y_{m_p}(t)\right) dt$$

Proceeding to limits as $p \to \infty$,

$$y_{m_p}(x) = y_0 + (x - x_0)y_1 + \int_{x_0}^{x} (x - t) f(t, y(t)) dt - \lim_{p \to \infty} \int_{x - \frac{a}{m_p}}^{x} (x - t) f\left(t, y_{m_p}(t)\right) dt$$

We now show that,

$$\lim_{p \to \infty} \int_{x - \frac{a}{m_p}}^{x} (x - t) f\left(t, y_{m_p}(t)\right) dt = 0$$

Now,

$$\left| \int_{x - \frac{a}{m_p}}^{x} (x - t) f\left(t, y_{m_p}(t)\right) dt \right| \le M \int_{x - \frac{a}{m_p}}^{x} |t - x| dt$$

$$= \frac{M}{2} \left(\frac{a}{m_p}\right)^2$$

$$\le \frac{M}{2} \left(\frac{a}{p}\right)^2$$

Therefore,

$$0 \le \lim_{p \to \infty} \left| \int_{x - \frac{a}{m_p}}^{x} (x - t) f\left(t, y_{m_p}(t)\right) dt \right| \le \lim_{p \to \infty} \frac{M}{2} \left(\frac{a}{p}\right)^2 = 0.$$

Therefore, by sandwich lemma,

$$\lim_{p \to \infty} \int_{x - \frac{a}{m_p}}^{x} (x - t) f\left(t, y_{m_p}(t)\right) dt = 0$$

With these observations, we have obtained equation (4), showing that $y(x)$ is indeed a solution of (3).

We can also locally extend Peano's theorem as:

Theorem 4 Let $f(x, y)$ be continuous and bounded in the rectangle in $\overline{S}: |x - x_0| \le a, \ |y - y_0| \le b,$ and hence $\exists\, M > 0,$ such that, $|f(x, y) \le M, \forall\, (x, y) \in \overline{S}.$ Then, IVP (2) has at least one solution in the interval $J_h: |x - x_0| \le h = \min\left\{a, \frac{b}{M_1}\right\}$ $where\ M_1 = |y_1| + \frac{Ma}{2}$

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