On Pseudo-Small-Projective Modules And Quasi Pseudo Weakly Projetive Module

Sushma Jat[1], Vivek Patel[2]

Abstract- *Pseudo-projectivity and small-pseudo-projectivity is a generalization of projectivity. In this paper the concept of small projectivity has been generalized to small pseudo projectivity and some results on small pseudo projective modules .Aiso some results on pseudo weakly projective modules have been obtained. Here the basic ring R is supposed to be ring with unity and all modules are supposed to be unitary left R-modules.*

Keywords- small pseudo projective modules, weakly projective, projective cover.

I. INTRODUCTION

Throughout this paper all rings are associative rings with identity, and all modules are unitary right modules. Pseudo-injective and essential-pseudo-injective modules have been studied by many authors [2], [3], but here we study the properties of pseudo and small pseudo-projective modules.

Suppose that M is an R-module. A submodule A of M is said to be a small submodule of M if for any B ⊆ M, A + B = M implies B = M. Given two R-modules N and M,
N is called M-projective if for every submodule A of M, any homomorphism α : N → M/A can be lifted to a homomorphism β : N → M. A module N is called projective if it is M-projective for every R-module M. On the other hand, N is called quasi-projective if N is N-projective.

It is well-known that an R-module N is projective if and only if it satisfies any of the following equivalent conditions:

i) For any R-module M and B ⊆ M every R-homomorphism N → M/B can be lifted to an R-homomorphism N → M.

ii) For any R-module M, every R-epimorphism M → N splits.
Moreover, we will denote projective cover ofM by (P(M), αM) if, there is an epimorphism α_M : P(M) → M with P(M) projective and ker(α_M) $\ll$ P(M).

Theorem 1.1 Let M and N be two modules and X = M ⊕ N. Then following conditions are equivalent:
(i) N is small-M-pseudo-projective;
(ii) for any submodule A of X such that N is a supplement of A and A+M =X, there exists a submodule N'⊆ A such that N' ⊕M = X.

Proof

(i) =⇒ (ii) Assume that holds i) and A satisfies the assumptions of (ii) and also let
f : N → M/(A ∩ M). If n ∈ N then there exists m ∈ M, a ∈ A such that n = a + m. Define
f(n) = m + A ∩ M. If n1 = n2 in Nthen n1 = a1 + m1 and n2 = a2 + m2 for some a1, a2 in A and m1,m2 in M.Then a1 − a2 = m2 − m1 is in A ∩ M. Thus f(n1) = f(n2). Clearly f is an epimorphism. ker(f) = A ∩ N _ N. As N is small M-pseudo-projective,f can be lifted to a homomorphism f_ : N → M such that π ◦ f_ = f, withπ : M → M/(M ∩ A). Define N' = {n – f'(n)|n ∈ N} and let x ∈ N', thenx = n – f'(n) for some n ∈ N. We have π ◦ f'(n) = f(n) and n = a + m
for some a in A and m in M. Therefore f'(n) + A ∩M = m + A ∩M. Thenf'(n)−m is in A∩M and f'(n)−n+a is in A∩M. Thus x ∈ A. Then N' ⊆ A.It is clear that N' +M = X. Also N' ∩M = 0. Consequently N'⊕M = X.(ii) =⇒ (i) Let f : N → M/B be an epimorphism with ker(f) $\ll$ N and
π : M → M/B is natural projection. Define A = {n + m|f(n) = −π(m)}. Itis clear that X = N +A and A∩N = ker(f) $\ll$ N and X = A+M, then thereexists a submodule N' ⊆ A with X = N' ⊕M. Define α : N' ⊕M → M withα(n_ + m) = m Then $\alpha|_N$: N → M is lifting homomorphism of f. ThereforeN is M-pseudo-projective.

Theorem.1.2 If N be a small-M-pseudo projective module and B be adirect summand of M, then N is small-B-pseudo projective module.

proof. Let X' = N ⊕B and X = N ⊕M. Assume that A ⊆ X' such that N is supplement of A inX' and X = N ⊕ B. Then we have N is supplement of Ain X andA + B = X. By Proposition (3.1) there exists a submodule N_ ⊆ A such that N_ ⊕M = X. Then N_ ⊕ B = X_. Again by Proposition if N is M-pseudo-projective then any epimorphism f : M →N splits.N is small-B-pseudo projective.

Theorem1.3 Let M be a small pseudo projective module and φ : M → N be a small epimorphism then there exists an epi-endomorphism h in End(M) such that Kerφ = Ker(φ o h) is stable under h.

Proof : The small epimorphism φ : M → N induces an isomorphism.φ* :M / Kerφ → N
Let f : M →M/Kerφ be the natural map.
By small pseudo projectivity of M ∃h∈End (M) such thatφ* of =φ o h .
Since φ (M) = N =φ *of (M) =φ o h(M);
for any m∈M, φ (m) =φ o h(m)
⇒m-h (m) ∈Kerφ ⇒ m∈Imh + Kerφ
⇒M⊆ Imh + Kerφ ⇒ M = Imh + Kerφ ⇒ M= Im h,
⇒ h is onto.
Now, x∈Kerφ ⇒ x∈ Ker f ⇒ f (x) = 0
⇒φ *of (x) = 0⇒φ oh (x) = 0
⇒ x∈ Kerφ o h⇒ Kerφ ⊆ Kerφ o h
Let y ∈Kerφ oh
⇒φ o h (y) = 0⇒φ *of (y) = 0
⇒ f (y)∈ Kerφ*⇒ f (y) = o asφ *is one −one
⇒ y ∈ Ker f = Kerφ ⇒ Kerφo h ⊆ Kerφ
and therefore Ker φ = Kerφ oh
Now let, t ∈Kerφ then φ (t) = 0 = φ o h (t) ⇒ φ {t – h (t)} = 0
⇒t − h(t)∈ Kerφ ⇒h(t)∈Kerφ ⇒h(Kerφ) ⊆ Kerφ .
Hence Kerφ is invariant under epi-endomorphism of M.

Theorem1.4 Let M be a small pseudo projective module and K⊆ M be a small submodule of M then M/K is small pseudo projective if K is stable under endomorphisms of M.

Proof: Let v **:** M→M/K be the natural map, f : M/K →A be an epimorphism and g: M/K→A be a small epimorphism, where A is any R- module. Then by small pseudo projectivity of M there exist φ ∈ End (M) such that f o v = go v oφ .

Define

$$\psi: \frac{M}{K} \to \frac{M}{K}$$
$$as\ \psi(x+K) = \varphi(x) + K$$

Then ψ is well defined and

ψ o v = v oφ ⇒ g oψ o v = g o v oφ
⇒g oψ o v = f o v ⇒ g oψ = f , since v is onto
⇒ M/K is small pseudo projective.

II. QUASI PSEUDO WEAKLY PROJECTIVE

We say that M is quasi pseudo Projective If M has a projective cover $\pi: P(M) \to M$ and every homomorphism $\Psi: P(M) \to N$ can be factored through M via some epimorphism. Equivalently, a module M is weakly projective if it has a projective cover $\pi: P(M) \to M$ and given any homomorphism $\Psi: P(M) \to N$ there exists $X \subseteq$ ker Ψ such that $\frac{P(M)}{X} \cong M.$

Theorem2.12 Let M and N are two R-modules and assume M has a projective cover π : P ⟶ M. Then the following statements are equivalent :

1. M is quasi pseudo weakly projective.
2. For every sub module K ⊂ N, M is quasi pseudo weakly projective.
3. For every sub module K⊂ N, M is quasi pseudo weakly $\frac{N}{K}$-projective.

Proof :

(i) (1) Implies (2) and (3) Assume M is pseudo weakly N-projective and let K is a sub-module of N and ψ : P ⟶K is a homomorphism. Then $\psi = i_\pi$. ψ : P ⟶N may be expressed as a composition ψ=gσ for some homomorphism g : M ⟶ N and epimorphism σ : P⟶ M. Since σ is onto, the range of σ equals the range of g and so it is contained in K. Thus we may define g : M⟶ K via ψ(m) = g(m) and then ψ=gσ, proving that M is pseudo weakly K-projective as claimed. Assume once again that M is pseudo weakly projective and let f : P⟶ N/K is a homomorphism. Since P is projective, there exists a map $\bar{f}: P \longrightarrow N$ such that $f = \pi_x.\bar{f}$. The weakly N-projective of M yields an epimorphism σ :P ⟶M and a homomorphism h:M⟶N such that $\bar{f} = h.\sigma$. Let $\pi_x.h = f_1$ then $f_1\sigma = \pi_k.h.\sigma = \pi\bar{f}$ =f, proving that M is indeed pseudo weakly $\frac{N}{K}$ projective.
(ii) (2) or (3) implies (1) is trivially.

Remarks :

Let M and N are two R-modules and assume M has a projective cover π : P⟶ M. Then M is quasi pseudo weakly N-projective if and only if for every sub-module K⊂N and for every epimorphism ψ : P ⟶K there exist epimorphism σ : P ⟶ M and g : M ⟶ N such that ψ = g.σ.

Theorem:2.2

Let M and N are two R-modules and assume M has a projective cover $\pi : P \longrightarrow M$. Then M is quasi pseudo weakly N-projective iff for every map $\psi : P \longrightarrow N$ there exist a sub module $X \subset Ker\psi$ such that $\frac{P}{X} \cong M.$

Proof :Necessary condition :

Let $\psi : P \longrightarrow N$ is a homomorphism. Assume M is quasi pseudo weakly N-projective and let the homomorphism $g : M \longrightarrow N$ and the epimorphism $\sigma : P \longrightarrow M$ be as in the definition of weakly relative projectivity. Since $\psi = g.\sigma$, Ker $\sigma \subset$ Ker ψ. Also $\frac{P}{Ker\sigma} \cong M$ Thus the implication is proven by choosing X = Ker σ.

Conversely :

If $X \subset P$ satisfies the condition in the statement of the theorem, then the isomorphism $\frac{P}{X} \cong M$, composed with the natural projection $\pi_k : P \longrightarrow P/X$ is an epimorphism $\sigma : P \longrightarrow M$ satisfying that Ker $\sigma = X \subset$ Ker ψ. It follows that the map $g : M \longrightarrow N$ given by $g(m)= \psi(\rho)$ whenever $\sigma(\rho)=M$ is well defined and satisfies $\psi=g.\sigma$

REFERENCES

[1] A.K. Tiwary and B.M. Pandeya Pseudo Projective and Pseudo InjectiveModules, Indian J. of pure appl. Math., Vol. 9, No. 9(1978), pp. 941-949.

[2] A.K. Tiwary, P.C. Bharadwaj and B.M. Pandeya, Quotients of PseudoProjective Modules, Bulletin of the Inst. of Math. , Academia Sinica,Vol.10, No.4, December (1982), 357-367.

[3] K.N. Chaubey: Small Projective Modules, Ph.D. Thesis, Banaras Hindu University, India (1984).

[4] Dinguuo Wang and Jinqui Li: Characterization of rings using weakly projective module. Vietnam journal of mathematics.vol.25.(1998)

[5] Quasi weakly projective and quasi weakly injective module: M.R. Aloney S. Jat. Ultra scientist journal of physical sciences.(2011)

[6] A.K.Tiwary and B.M.Pandeya: Pseudo projective and pseudo injective modules. B.H.U. Vol.9 No.9(1978)