Characterizations of Secondary Hadamard Matrices

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Abstract*-The concept of s-Hadamard matrices is introduced. Characterization of s-Hadamard matrices are obtained and derived some theorems. Theorems relating to s-Hadamard Matrices and s-Orthogonal Matrices are also derived.*

Keywords-Hadamard matrix, s-Hadamard matrix, Secondary. transpose of the matrix, s-orthogonal matrix

I. INTRODUCTION

Anna Lee **[1]** has initiated the study of secondary symmetric matrices. Also she has shown that for a complex matrix A, the usual transpose A^T and secondary transpose A^S are related as $A^s=VA^TV$ **[2]** where 'V' is the permutation matrix with units in its secondary diagonal. James Sylvester introduced Hadamard matrices in 1867 as square matrix with entries ±1 which satisfies $HH^T = nI$**[3]** where H^T is the transpose of H and I is the identity matrix..Number of applications of Hadamard matrices to signal processing, optical multiplexing, error correction coding, and design and analysis of statistics given by HaralambosEvangelaras, Christos Koukouvinos, and Jennifer Seberry**[6].**The new series of Hadamard matrices is constructed by Masahiko Miyamoto**[7].**Hadmard product of Hadamard matrices was given by Elizabeth million**[4].**GeramitaA.V.andseberryJ.gave the concept of Orthogonal Designs, Quadratic forms and Hadamard matrices**[5].**

In this paper the s-Hadamard matrices is defined and its characterizations are discussed. Some theorems relating to s-Hadamard Matrices and s-Orthogonal Matrices are derived.

II. PRELIMINARIES AND NOTATIONS

Let $C_{n\times n}$be the square of n×n complex matrices of order n. For $A\in C_{n\times n}$.A^T, $\overline{A}$,A^*,A^sdenote transpose, conjugate, conjugate transpose, secondary transpose of a matrix A respectively. Let 'V' be the associated permutation matrix whose elements on the secondary diagonal are 1 and other elements are zero. Also 'V' satisfies the following properties. $V^T=V^{-1}=\overline{V}=V^*=V$ and $V^2=I$.

A matrix $A\in C_{n\times n}$ is called Hadamard matrix if $HH^T= nI$ where n is the order of a matrix.

III. SECONDARY HADAMARD MATRICES

Definition 3.1:
Let $A\in C_{n\times n}$.A is said to be secondary Hadamard matrix or s-Hadamard matrix if $AA^TV= nV$ where A^T is the transpose of A.
The matrix A is called s-orthogonal, if $AA^s = A^sA = I$. That is $A^T V A = V$ **[8]**

IV. PROPERTIES OF S-HADAMARD MATRIX

Theorem 4.1:
Let $A\in C_{n\times n}$. If A is a s-Hadamard matrix then A^T is also s-Hadamard matrix.
Proof:
We know that if A is a s-Hadamard matrix then $AA^T V=nV$
Taking transpose on both sides
$(AA^TV)^T= (nv)^T$
$\Rightarrow V^T(A^T)^TA^T= nV^T$
$\Rightarrow VAA^T=nV$
i.e., $AA^TV= nV$
Therefore$A^T$is a s-Hadamard matrix.

Theorem 4.2:
Let $A\in C_{n\times n}$.If A is a s-Hadamard matrix then AA^T is also s-Hadamard matrix.

Proof:
We know that if A is a s-Hadamard matrix then $AA^T V=nV$
Claim: $(AA^T)(AA^T)^TV=nV$
$\Rightarrow(AA^T)(AA^T)^TV = (AA^T)(A^T)^TA^TV$
$\Rightarrow(AA^T)(AA^T)^TV = AA^TAA^TV$
$\Rightarrow(AA^T)(AA^T)^TV = nVnVV$
$\Rightarrow(AA^T)(AA^T)^TV= nV$
Therefore $AA^T$is a s-Hadamard matrix.

Theorem 4.3:
Let $A\in C_{n\times n}$. If A is a s-Hadamard matrix then AV is s-Hadamard matrix.
Proof:
We know that if A is a s-Hadamard matrix then $AA^T V=nV$
Claim: AV is s-Hadamard matrix
i.e., $(AV)(AV)^TV=nV$
pre-multiply by V on both sides,we get
$VAA^TV=VnV$

$\Rightarrow VAA^TV=nV^2$
$\Rightarrow V^TAA^TV=n$
$\Rightarrow (A^TV^T)(AV)=n$
$\Rightarrow (AV)^TAV=n$
$\Rightarrow (AV)^T(AV)V=nV$
$\Rightarrow (AV)(AV)^TV = nV$
Therefore AV is s-Hadamard matrix

Theorem 4.4
Let $A\in C_{n\times n}$. If A is a s-Hadamard matrix then VA is s-Hadamard matrix.
Proof:
If VA is s-Hadamard matrix then we have to prove $(VA)(VA)^TV=nV$
We know that $AA^TV=nV$
Post-multiply by V on both sides
$AA^TVV=nVV$
$\Rightarrow AA^TVV=nV^2$
$\Rightarrow AA^TVV=nI$
$\Rightarrow AA^TV^TV=nI$
$\Rightarrow A(VA)^TV=nI$
$\Rightarrow (VA)(VA)^T= Vn= nV$
Therefore VA is s-Hadamard matrix.

Theorem 4.5:
Let $A\in C_{n\times n}$. If A and B are s-Hadamard matrix then AB is s-Hadamard matrix .
Proof:
If A is a s-Hadamard matrix then $AA^TV=nV$
If B is a s-Hadamard matrix then $BB^TV=nv$
Claim:$(AB)(AB)^TV=nV$
$(AB)(AB)^TV=B^TA^T(AB)V$
$\Rightarrow (AB)(AB)^TV=B^T(AA^TV)B$
$\Rightarrow (AB)(AB)^TV=B^T(nV)B$
$\Rightarrow (AB)(AB)^TV=nBB^TV$
$\Rightarrow (AB)(AB)^TV=n(nV)$
$\Rightarrow (AB)(AB)^TV=nV$
Therefore AB is s-Hadamard matrix .

Theorem 4.6:
Let $A\in C_{n\times n}$. If A and B are s-Hadamard matrix and if $BA^TV + AB^TV = -nV$then A+B is s-Hadamard matrix .
Proof:
If A is a s-Hadamard matrix then $AA^TV=nV$
If B is a s-Hadamard matrix then $BB^TV=nv$
Claim:$(A+B)(A+B)^TV=nV$
$\Rightarrow (A+B)(A+B)^TV = (A+B)(A^T+B^T)V$
$\Rightarrow (A+B)(A+B)^TV = (AA^T+BA^T+AB^T+BB^T)V$
$\Rightarrow (A+B)(A+B)^TV = nV-nV+nV$
$\Rightarrow (A+B)(A+B)^TV = nV$
Therefore if A and B are s-Hadamard matrices and $BA^TV+AB^TV=-nV$ then A+B is s-Hadamard matrix.

V. RELATION BETWEEN S-HADAMARD MATRICES AND S-ORTHOGONAL MATRICES

Theorem 5.1:
Let $A\in C_{n\times n}$. If A is a s-Hadamard matrix then A^{-1} is also s-orthogonal matrix.
Proof:
We know that if A is s-orthogonal matrix. then $AA^S = A^SA = I$
Claim: $(A^{-1})(A^{-1})^S=I$
$AA^s=I$
Taking inverse on both sides
$(AA^s)^{-1} = (I)^{-1}$
$\Rightarrow (A^s)^{-1}A^{-1} = I^{-1}$
$\Rightarrow (A^{-1})^sA^{-1} = I^{-1}$
Then $(A^{-1})(A^{-1})^s = I$
Similarly we may prove $(A^{-1})^s(A^{-1}) = I$
Therefore A^{-1} is s-orthogonal matrix.

Theorem 5.2:
Let $A\in C_{n\times n}$. If A is a s-Hadamard matrix then A* is s-orthogonal matrix.

Proof:
We know that if A is s-orthogonal matrix then $AA^S = A^SA = I$
Claim: $A^*(A^*)^s=I$
Taking conjugate transpose on both sides
$(AA^s)^*=(I)^*$
$\Rightarrow (A^s)^*A^*=I^*$
$\Rightarrow (A^*)^sA^*=I$
Similarly we may prove $A^* (A^*)^s=I$
Therefore A* is a s-orthogonal matrix.

Theorem 5.3:
Let $A\in C_{n\times n}$.If A is a s-Hadamard matrix then A^S is s-orthogonal matrix.
Proof:
We know that if A is s-orthogonal matrix then $AA^S = A^SA = I$
Claim: $A^S(A^S)^S=I$
Taking secondary transpose on both sides
$\Rightarrow (AA^S)^S=I^S$
$\Rightarrow (A^S)^SA^S=I$
$\Rightarrow A^S(A^S)^S=I$
Similarly we may prove $(A^S)^S A^S = I$
Therefore A^S is s-orthogonal matrix.

VI. CONCLUSION

In this paper the concept of s-Hadamard matrices was defined and theorem relating to characterizations of s-Hadamard matrices were derived. Theorems relating to s-Hadamard Matrices and s-Orthogonal Matrices were also derived.

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