# Some Domination Related Results on Directed Graphs Analogous To Those of Undirected Graphs

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***Abstract-*** *This paper provides some fundamental results on domination and domination number for various directed graphs.. We explore some domination related results on digraphs analogous to those of undirected graphs and also find the bounds for digraphs*

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## I. INTRODUCTION

Domination and other related concepts in undirected graphs are well studied. The pioneering work in digraphs in this area can be ascribed to Berge, Harary, Konig, Grundy and Richardson, amongst others..In this chapter some results on domination in digraphs; the concept and results concerning solutions in a digraph and the application of some of these ideas to game theory. The significant works in these areas by Blidia, Duchet, Galeana-Sanchez, Kwasnik, Meyneil, Neumann-Lara, Roth, Smith, Topp and others are recorded in this endeavor. We begin this journey with definitions of the major concepts.

### 1.2 Definitions

Perhaps no other area of domination has as great a need to standardize definitions and notation as that of directed domination. Different terms are chosen for the same concept and the same term is occasionally chosen for different concepts. We have tried to clarify the situation by giving common alternate terms and pointing out differences in definitions. For this paper, unless otherwise mentioned, a graph D = (V, A) consists of a finite vertex set V and an arc set A ⊆P, where P is the set of all ordered pairs of distinct vertices of V. That is, D has no multiple loops and no multiple arcs (but pairs of opposite arcs are allowed). For this paper we assume that the underlying graph of the digraph D is connected. In the terminology of Berge we are considering connected 1-graphs without loops. Let D = (V, A) be such a digraph. If A = P then the digraph is complete. Following Berge, a subset S ⊆ V is absorbant if for every vertex x ≠ S there is a vertex y ∈ S such that y is a successor of x. We define a set S ⊆ V of a digraph D to be a dominating.

### 1.3 Definition

A directed graph (also called a digraph) D=(V.A) consists of a finite non empty set V of vertices and a finite non empty set A of directed edges called arcs, where $A \subseteq V \times V - \{(u,u) \mid u \in V\}$. An arc $(u,v)$ is said to be directed from $u$ to $v$. We also say that $u$ is adjacent to $v$ or $v$ is adjacent from $u$. In this case we use the notational equivalence $(u,v) = u \longrightarrow v$. Also $(u,v) = uv$. The vertex $u$ is called a predecessor of $v$ and $v$ is called a successor of $u$. $u$ is the head and $v$ is the tail of the arc $(u,v)$. If the reversal $(v,u)$ of an arc $(u,v)$ of D is also present in D we say that $(u,v)$ is a reversible (symmetric) arc. If $(u,v) \in A$ but $(v,u) \notin A$ then $(u,v)$ is an asymmetric arc.

### 1.4 Domination in Digraphs

Although the concept of domination in graphs has received extensive attention as evidenced by this volume, the same concept has been somewhat sparsely studied for digraphs. Even bounds undirected graphs have not been considered and compared with their counterparts for digraphs. In terms of applications, the questions of Facility Location, Assignment Problems etc. are very much related to the idea, of domination or independent domination on digraph. There have been over the year's a few papers on the domination number of digraphs. These and other related concepts are presented below. We use the notation $\gamma$(D) to represent the domination number of a digraph, i.e., the minimum cardinality of a set S $\subseteq$V(D) which is dominating

**Definition 1.4.1**

Let $D = (V, A)$ be a digraph. Let $u, v \in V$. We say that $u$ dominates $v$ if $(u, v) \in A$.

We observe that in the case of digraphs the relation 'dominates' is not symmetric. In the digraph $D$ given in Fig. 1, 1 dominates 2, but 2 does not dominate 1.

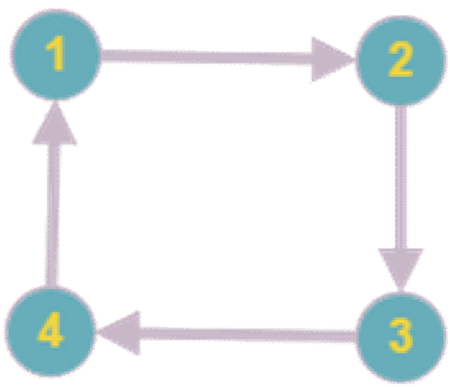


Fig. 1

**Definition 1.4.2**

Let $D = (V, A)$ be a digraph. A subset $S$ of $V$ is called a dominating set of $D$ if for every vertex $v \in V - S$, there exists a vertex $u \in S$ such that $u$ dominates $v$. The domination number $\gamma(D)$ of $D$ is the minimum cardinality of a dominating set in $D$.

When there is no possibility of confusion we denote $\gamma(D)$ by $\gamma$. A dominating set of cardinality $\gamma$ is called a γ - set or a minimum dominating set.

**Example 1.4.3**

(i) For the digraph $D$, given in Fig. 1, $S = \{1, 2\}$ is a dominating set. Further there is no dominating set of cardinality 1 for $D$ and hence $\gamma(D) = 2$

(ii) Consider the digraph $D$ given in Fig. 2 $S = \{1\}$

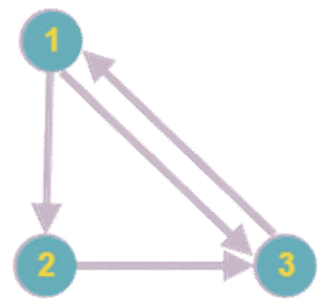


Fig. 2

Is a dominating set of $D$ and hence $\gamma(D) = 1$.

(iii) Consider the directed path $P_n = (1, 2, \dots \dots n)$. Then $S = \{i / I \leq i \leq n$ and $i$ is odd$\}$ is a dominating set of $P_n$ and $|S| = \left\lceil \frac{n}{2} \right\rceil$. Hence $\gamma(P_n) \leq \left\lceil \frac{n}{2} \right\rceil$. Further each vertex $i$ of $P_n$ with $i \neq n$ dominates exactly one vertex namely $i + 1$ and hence it follows that $\gamma(P_n) \geq \left\lceil \frac{n}{2} \right\rceil$. Thus $\gamma(P_n) = \left\lceil \frac{n}{2} \right\rceil$.

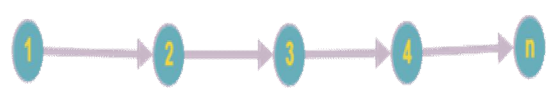


Fig. 3

Pn

Similarly for the directed cycle $C_n = (1, 2, \dots \dots \dots n, 1)$ we have $\gamma(C_n) = \left\lceil \frac{n}{2} \right\rceil$

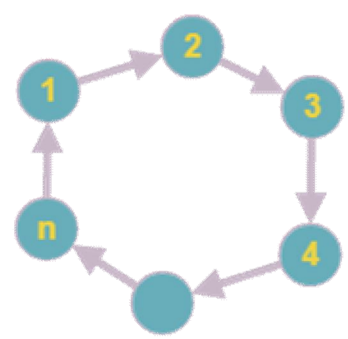


Fig. 4

(iv) The domination number of a symmetric digraph is equal to the domination number of its underlying graph.

(v) The domination number of a complete symmetric digraph is 1.

(vi) For the digraph $D$ given in Fig. 5, $S = \{3,4\}$ is a minimum dominating set.

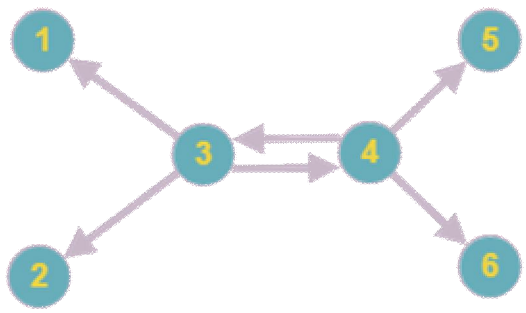


Fig. 5

And hence $\gamma(D) = 2$ and for its converse $S = \{1,2,5,6\}$ is a minimum dominating set and so $\gamma(D^1) = 4$.

(vi) For the digraph $D$ given in Fig. 6, $S = \{2,3,4\}$ is a minimum dominating set so that $\gamma(D) = 3$ and for $D', S = \{1,5,6\}$ is a minimum dominating set and hence $\gamma(D^{-1}) = \gamma(D) = 1$.

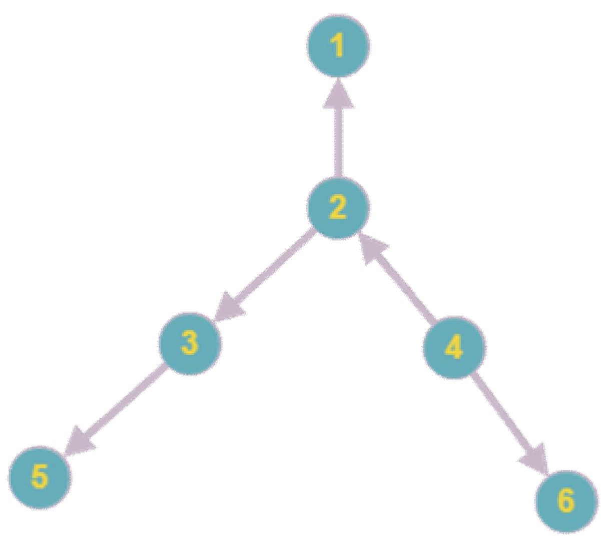


Fig. 6

**Remark 1.4.4**

The following are equivalent ways of looking at a dominating set in digraphs.

Let $D = (V,A)$ be a digraph. A subset $S$ of $V$ is a dominating set if and only if any one of the following is true.

1. For every vertex $v \in V - S$, there exists a vertex $u \in S$ such that $v$ is adjacent from $u$.

2. For any vertex $v \in V - S, d(S,v) \leq 1$.

3. $O[S] = V$

4. For every vertex $v \in V - S, |I(v) \cap S| \geq 1$.

5. For every $v \in V - S, v$ is a successor of some vertex in $S$.

**Remark 1.4.5**

Let $D = (V,A)$ be a digraph. If $v$ is a vertex with $id(v) = 0$, then $v$ lies in every dominating set of $D$.

**Remark 1.4.6** $\gamma(D) = 1$ if and only if there exists a vertex $v$ in $D$ such that $od(v) = n - 1$

**Remark 1.4.7**

Berge [6] introduced the concept of kernel in digraphs. Let $D = (V,A)$ be a digraph. A subset $S$ of $V$ is said to be an absorbant if every $v \in V - S$ dominates at least one vertex in $S$. Thus $S$ is an absorbant if and only if $S$ is a dominating set of $D^{-1}$ where $D^{-1}$ is the converse of the digraph $D$. A subset $S$ of $V$ is independent if no two vertices of $S$ are joined by an arc. $S$ is called a kernel of $D$ if it is both independent and absorbant. Most of the research papers in domination in digraphs deal with the existence of kernels in digraphs ([10], [2], [14])

**Definition 1.4.8**

Let $D = (V,A)$ be a digraph. A subset $S$ of $V$ is called a minimal dominating set of $D$ if $S$ is a dominating set of $D$ and no proper subset of $S$ is a dominating set of $D$.

**Remark 1.4.9**

We observe that any minimum dominating set of $D$ is a minimal dominating set. However the converse is not true. Consider the digraph $D$ given in Fig. 7

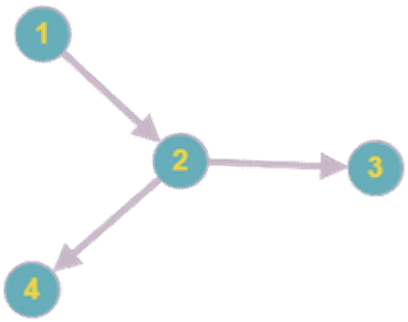


Fig. 7

$S = \{1, 3, 4\}$ is a minimal dominating set of $D$. However $\gamma(D) = 2$ and $\{1, 2\}$ is a minimum dominating set.

Ore [16] has obtained a necessary and sufficient condition for a dominating set of a graph to be minimal dominating set. The following theorem gives the analogous result for digraphs.

**Theorem 1.4.10**

A dominating set $S$ of a digraph $D$ is a minimal dominating set if and only if for each vertex $u \in S$, one of the following conditions holds.

a) $u$ is not a recipient of $S$.

b) There exists a vertex $v \in V - S$ for which $I(v) \cap S = \{u\}$.

**Proof**

Assume that $S$ is a minimal dominating set of the digraph $D$. Then for any vertex $u \in S, S - \{u\}$ is not a dominating set of $D$. Hence there exists a vertex $v$ in $(V - S) - \{u\}$ such that $v$ is not dominated by any vertex in $S - (u)$. Now either $v - u$ or $v \in V - S$. If $v, u$ then $u$ is not a recipient of $S$. If $v \in V - S$, then $v$ is not dominated by $S - \{u\}$, but it is dominated by $S$. Hence $v$ is adjacent only from $u$ in $S$, so that $I(v) \cap S = \{u\}$.

Conversely suppose that $S$ is a dominating set and for each $u \in S$, one of the two conditions holds. Suppose $S$ is not a minimal dominating set. Then there exists a vertex $u \in S$ such that $S - \{u\}$ is a dominating set. Hence $u$ is adjacent from at least one vertex of $S - \{u\}$, so that $u$ is a recipient of $S$. Also every vertex in $V - S$ is adjacent from at least one vertex in $S - \{u\}$. Thus neither condition (a) nor (b) holds, which is a contradiction. Hence $S$ is a minimal dominating set of $D$.

**Remarks 1.4.11**

Let $G = (V, E)$ be a graph without isolated vertices. If $S$ is a minimal dominating set of $G$, then $V - S$ is also a dominating set of $G$ and hence $\gamma(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$. The following example shows that a similar result is not true for digraphs even if the indegree and outdegree of every vertex is greater than zero. For the digraph $D$ given in Fig. 8, $S = \{1, 2\}$ is a minimal dominating set and $V - S = \{3\}$ is not a dominating set and $\gamma(D) = 2 > \left\lfloor \frac{3}{2} \right\rfloor$.

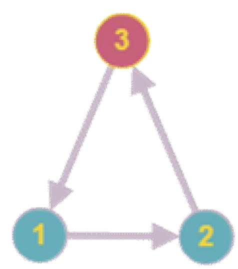


Fig. 8

**Definition 1.4.12**

Let $D = (V, A)$ be a digraph. The maximum cardinality of a minimal dominating set of $D$ is called the upper domination number of $D$ and is denoted by $\Gamma(D)$ or simply $\Gamma$, when there is no possibility of confusion.

It follows immediately from the definition that $\gamma(D) \leq \Gamma(D)$. For the digraph given in Fig. 9, we have $\gamma = 3$ and $\Gamma = 5$.

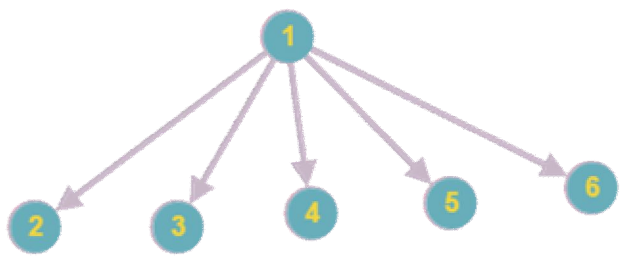


Fig. 9

**Remark 1.4.13**

A dominating set $S$ is minimal dominating if and only if for every $v \in S$, there exists a vertex $w \in V - (S - \{v\})$ such that $w$ is not dominated by $S - (v)$.

**1.5. Result and Analysis**

Now we explore some domination related results on digraphs analogous to those of undirected graphs. First we look at some common bounds for γ(D). One of the earliest bounds for the domination number for any undirected graph was proposed by Ore[16].

**Theorem 1.5.1** For any graph G without isolates, $\gamma(G) \le \frac{n}{2}$, where n is the number of vertices Ore[16] .

This result does not hold for directed graphs; a counter example is the digraph $K_{1,n}$, $n \ge 2$, with its arcs directed from the end vertices towards the central vertex. The general bound which holds for digraphs is not very good for a majority of digraphs. We assume our digraphs to be those whose underlying graphs are connected.

**Observation 1.5.2** For any digraph' with n vertices, $\gamma(D) \le n - 1$.

This bound is sharp because the domination number of the digraph $K_{1,n}$ for $n \ge 2$ with its arcs directed from the end vertices towards the central vertex is n. Since very few graphs agree with this bound we find other bounds which are tighter for a significant number of digraphs.

**Theorem 1.5.3** For any digraph D on n vertices, $\frac{n}{1+\Delta(D)} \le \gamma(D) \le n - \Delta(D)$, where $\Delta(D)$ denotes the maximum outdegree.

**Proof.** For the upper bound we form a dominating set of D by including the vertex υ of maximum outdegree and all the other vertices in the digraph which are not dominated by v. This set is clearly a dominating set and has cardinality $n - \Delta(D)$.
Note that any vertex in D can dominate at most $1 + \Delta(D)$ vertices. In a minimum dominating set S of D there are γ(D) vertices, so they can dominate at most $\gamma(D)(1 + \Delta(D))$ vertices. Since S is dominating this number has to be at least n. Thus we get the lower bound.
To get another bound we look for certain characteristics in a digraph.

**Observation 1.5.4** For any digraph D on n vertices, which has a hamiltonian circuit, $\gamma(D) \le [\frac{n}{2}]$

**Proof.** Let D contain a hamiltonian circuit C. To dominate the vertices of D it suffices to dominate the cycle C. We know that the domination number of a circuit is bounded above by $[\frac{n}{2}]$ and so the same holds for the digraph D.

**Theorem 1.5.5** For a strongly connected digraph D on n vertices, $\gamma(D) \le [\frac{n}{2}]$

In addition to γ(D) we introduce some domination related parameters in digraphs, in particular, the irredundance number, the upper irredundance number and the upper domination number, analogous to those for undirected graphs. Recall that a set $S \subseteq V(D)$ of a digraph D is a dominating set if for all $v \notin S$, v is a successor of some vertex in S. A dominating set S is a minimal dominating set if for every $v \in S$, $O[v] - O[S - v] \ne 0$. If $u \in O[\upsilon] - O[S - \upsilon]$, then u will be called a private outneighbor (pon) of υ with respect to S. See [38] for another characterization of minimal dominating sets in digraphs.
Let Γ(D), the upper domination пшпЬer, denote the maximum cardinality of a minimal dominating set. As in the undirected

case, we define an irredundant set $S \subseteq V(D)$ to be a set such that every $\upsilon \in S$ has a private out neighbor. The irredundance number ir(D) and the иppeт irredundance number IR(D) are, respectively, the minimum and maximum cardinalities of a maximal irredundant set.

The notion of a solution also yields parameters which are new to the field of digraphs. Let i(D) and $\beta$(D) denote respectively the minimum and maximum cardinalities of an independent dominating set. It must be pointed out that not all digraphs have independent dominating sets. As these are special cases of solutions, these exist in digraphs which admit at least one solution. It must be mentioned here that due to the concepts defined above the following string of inequalities hold for any digraph D with a solution,

$$r(D) \leq \gamma(D) \leq i(D) \leq \beta(D) \leq \Gamma(D) \leq IR(D).$$

Researchers interested in domination theory for undirected graphs are quite familiar with the corresponding inequality chain. This chain raises some interesting questions about the structural properties of digraphs D (having a solution), for which

1.$\gamma(D) = i(D)$,
2. $ir(D) = \gamma(D)$,
3.$\beta(D) = \Gamma(D) = IR(D)$, or
4. $i(D) \neq \beta(D)$.

The following theorem is an interesting result for transitive digraphs.

**Theorem 1.5.6** In a transitive digraph D, we have $\gamma(D) = i(D) = \beta(D)= \Gamma(D) = IR(D)$.

**Proof.** Note that if D is a transitive digraph so is its reversal $D^{-1}$. It is then known that a solution exists in D. Moreover, from Berge's theorem, we see that in D, every minimal absorbant set is independent and the kernel is unique. This implies that $\gamma(D) = I(D) = \beta(D)= \Gamma(D)$.

To show $\beta(D) = IR(D)$, suppose that S is an irredundant set with $|S| = IR(D)$. We will call such a set an IR-set. Amon.gst all IR-sets let S contain the minimum number of arcs in it. If S has no arcs, then certainly S is independent and $\beta(D) \geq IR(D)$ implying $\beta(D) = IR(D)$. So suppose that $< S >$ contains an arc (x, y). Since S is irredundant y must have a private outneighbor $y_1 \notin S$. But D is a transitive digraph, so $(x, y_1)$ must be an arc, contradicting that $y_1$ is a private neighbor of x. Hence S is independent and the result follows.

### 1.6. Applications

#### 1.6.1. Game Theory (Von Neumann, Morgenstern )

Suppose that *n* players, denoted by (1),(2),...,(n) can discuss together to select a point x from a set ***X*** (the "situations"). If player (i) prefers situation *a* to situation b, we shall write $a \geq^{I}$ b. The individual preferences might not be compatible, and consequently it is necessary to introduce the concept of effective *preference.* The situation *a* is said to be *effectively preferred to* b, or *a* > b, if there is a set of players who prefer *a* to b and who are all together capable of enforcing their preference for *a.* However, effective preference is not transitive; i.e., *a*> b and b > c does not necessarily imply that *a* > c.

Consider the digraph *D* = (V, A) where O(x) denotes the set of situations effectively preferred to x. Let S be a kernel of *D.* Von Neumann and Morgen-stern suggested that the selection be confined to the elements of S. Since S is independent, no situation in S is effectively preferred to any other situation in S. Since S is absorbant, for every situation $x \ni S$, there is a situation in S that is effectively preferred to x, so that x can be immediately discarded.

#### 1.6.2. Problem in Logic (Berge [6])

Let us consider a set of properties $P = \{p_1, p_2\}$ and a set of theorems of the type: "property $p_i$ implies property $p_j$' . These theorems can be represented by a directed graph D = (V, A) with vertex set *P*, where $(p_i, p_j)$ is an arc if and only if it follows from one or more of the existing theorems that $p_i$ implies $p_j$ Suppose we want to show that no arc of the complementary graph $\vec{D}$ is good to represent a true implication of that kind: more precisely, with each arc (p, q) with $p \neq q$ and $(p, q) \ni A$, we assign a student who has to find an example where p is fulfilled but not q (i.e., a counter-example to the statement that p implies q).

In [10] they determined the minimum number of students needed to show that all the possible (pairwise) implications are already represented in the di-graph D. It was found that this number corresponded to the cardinality of the unique kernel of the transitive digraph under study.

#### 1.6.3. Facility Location

Let *D* = (V, A) be a digraph where the vertices represent "locations" and there is an arc from location *u* to location *υ* if location *υ* can be "reachedΓ' from location *u.* Assume that each "location" has a weight associated with it which represents some parameter pertinent to the study.

Choose a subset of "locations" such that those outside the set have an arc incident from a member of the set, which means that all the "locations" can be "serviced' by the members of the set S. Let w(S) denote the sum of the weights of the members of S. The problem of finding such a set S such that w(S) is minimized. The relevant graph theoretic concept is that of directed domination.

**1.7. Conclusions and Open Problems**

Domination and other related topics in undirected graphs are extensively studied, both theoretically and algorithmically. However, the corresponding topics on digraphs have not received much attention, even though digraphs come up *more* naturally in modelling real world problems. With this view in mind, we have made an attempt to survey some of the existing results on domination related concepts on digraphs. We have also introduced some parameters on digraphs analogous to domination parameters on undirected graphs. As a matter of fact, it seems that almost all domination related problems on undirected graphs, if they make sense in digraphs, may be investigated. Algorithmic aspects of these problems on digraphs will be another good area, of research.

## II. ACKNOWLEDGEMENT

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