Perfect Coloring of Planar Graphs And Related Aspects

H. R. Bhapkar
Dept of mathematics
MIT School of Engineering, Loni Kalbhor, Pune

Abstract- *This paper defines a conjecture on the coloring of planar graph, known as the perfect coloring of planar graph. We intend to study the relation between semi perfect coloring and perfect coloring of graphs. Precisely, we prove this conjecture for few particular planar graphs.*

Keywords- Perfect Coloring, Semi perfect coloring.

I. INTRODUCTION

In this section, we present a brief survey of those results of graph theory, which we shall need shortly. The reader is referred to [7, 8, 10] for a fuller treatment of the subject.

1.1 Graphs: A graph G is an ordered pair (V (G), E (G)) where i) V(G) is a non empty finite set of elements , known as vertices. V (G) is known as vertex set. ii) E(G) is a family of unordered pairs (not necessarily distinct) of elements of V, known as edges of G. E(G) is known as Edge set. [7]

1.2 Planar Graph: A graph G is a planar graph if it is possible to represent it in the plane such that no two edges of the graph intersect except possibly at a vertex to which they are both incident. Any such drawing of planar graph G in a plane is a planar embedding of G.[8]

1.3 * isomorphism: Two graphs are said to be *isomorphic if their geometric duals are isomorphic. [4]

1.4 Four Color Map Problem: Planar map is a set of pairwise disjoint subsets of the plane, known as regions of the map. Two regions a map are adjacent if they have a common boundary that is not a corner. A vertex or point of a map is said to be corner if it is a common point of three or more regions. A coloring of a graph is an assignment of colors to its vertices (or regions) so that no two adjacent vertices (or regions) have the same color. The set of all vertices (or regions) with same color in graph, is called a color class. [1]

Theorem 1.1: (Four Color Map Theorem) Every planar map can be colored with four or fewer colors

The Four Color Conjecture was first stated 200 years ago and finally proved conclusively in 1976. The professor of mathematics, Augustus De Morgan (1806-71) and his friend William Rowan Hamilton studied this theorem and gave first proof. In 1879, Alfred Kempe, published a short paper on coloring of maps. He added some other ideas of coloring. In 1879, Alfred Kempe published this proof in the American Journal of Mathematics in simple versions. In 1980, Tait P.G. offered independent solution to this problem. After collaborating with John Koch on the problem of reducibility, in 1976, Kenneth Appel and Wolfgang Haken gave the complete proof to the four color conjecture by reducing the testing problem to an unavoidable set with 1936 configurations. Because of the computer based proof, many Mathematicians were not agreeing with this proof. However, many proofs written by different Mathematicians have been found to be faulty. So all we have been waiting for the simple proof of this theorem. [2, 3, 4]

This problem is stated by Douglas B. West. He has published it in his book. He state that *"The vertices and edges of a graph G can be colored with Δ(G) + 2 colors such that adjacent vertices have different colors, incident edges have different colors and incident edge and vertices have different colors."* [9]

This type of coloring is known as Semi Perfect Coloring. If α minimum number of colors are required to color any planar graph G by semi perfect coloring, then it is denoted by SPC (G) = α

1.5 HB Graph: A region or face R of a planar graph is said to be a pivot region of graph if all other regions of graph are adjacent to R. Every region of a complete graph on four vertices (K_4) is a pivot region. So K_4 has four pivot regions. [6]

The number of pivot regions of a planar graph is known as Pivot Region Number of that Graph. It is denoted by PRN (G). A planar graph is said to be HB graph if it has a pivot region. [6]

II. MAIN RESULTS

2.1 Open Problem on Coloring of Planar Graphs:

Conjecture: How many minimum colors will be required to color planar graph such that

1. ***Adjacent vertices have different colors.***
2. ***Incident edges have different colors.***
3. ***Adjacent regions have different colors.***
4. ***A region, boundary edges and boundary vertices of that region have different colors.***

If β number of colors are required to color any graph G by perfect coloring, then it is denoted by PC (G) = β. This type of coloring is known as Perfect Coloring. We have proved this open problem partially.

Theorem 2.1 *If G is any planar graph then SPC (G) ≤ PC (G).*

Proof: By the definitions of SPC (G) and PC (G), clearly SPC (G) is less than PC (G). Without loss of generality, assume that G is a rose1 graph with 3 edges as given below.

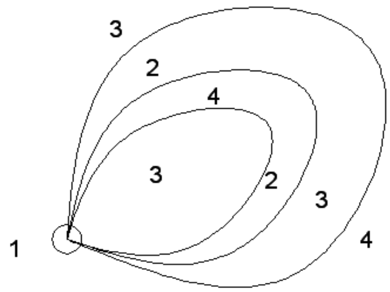

Figure 2.1: Rose 1 Graph

In a graph G, 3 edges are incident at only one vertex. So these edges are adjacent to each other. Therefore assign three different colors to these edges and one different color to the vertex. Thus SPC (G) = 4. Use same four colors only for coloring of regions with required conditions. So PC (G) = 4. Apply same judgment for rose 1 graph with any number of edges. This implies SPC (G) = PC (G). Thus SPC (G) ≤ PC (G). □

Theorem 2.2 *If G is a null graph with n vertices then SPC (G) = 1 and PC (G) =2.*

Proof: Let G be a null graph with n vertices say V_1, V_2, . . . V_n. Graph G has only one region R and all these n vertices lie in R. These vertices are totally disconnected. So assign same color to these vertices and different color to the region R. Thus SPC (G) = 1 and PC (G) = 2. □

Theorem 2.3 *If G is a chain graph on n ≥ 3 vertices then SPC (G) = Δ (G) + 1 and PC (G) = Δ (G) + 2, where Δ (G) = Highest degree of a vertex in G.*

Proof: Let G be a chain graph on n ≥ 2 vertices say V_1, V_2, . . . V_n, such that V_i is adjacent to V_{i+1}, for i = 1, 2, ... n-1. So V_1, V_n are pendent vertices. Assign color 1 to vertex V_1, color 2 to edge e_1 = {V_1, V_2} and color 3 to vertex V_2 in graph G. An edge e_2 = {V_2, V_3} is adjacent to e_1 and incident at V_2. So assign color to an edge e_2 different from colors 2 and 3. Therefore assign a color 1 to an edge e_2, color 2 to vertex V_3 and color 3 to an edge e_3 = {V_3, V_4}.

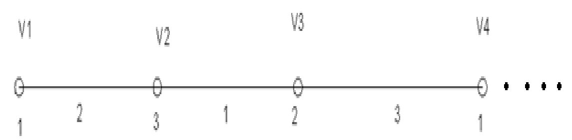

Figure 2.2: Chain Graph

The same sequence of colors is repeated finitely many times. Hence, 3 colors are required to color chain graph according to semi perfect coloring. Graph G has only one region. So assign a different color to this region. In chain graph G, Δ (G) = 2. Thus SPC (G) = 3 = Δ (G) + 1 and PC (G) = 4 = Δ (G) + 2. □

Theorem 2.4 *If G is a star graph on n ≥ 3 vertices then SPC (G) = Δ (G) + 1 and PC (G) = Δ (G) + 2, where Δ (G) = Highest degree of a vertex in G.*

Proof: Let G be a star graph on n vertices say V_1, V_2, . . . V_n. Subsequently, graph G has one vertex of degree n-1 and n-1pendent vertices.

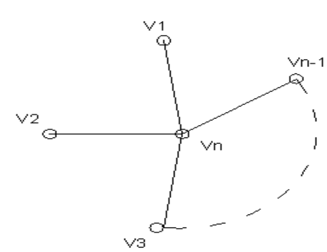

Figure 2.3: Star Graph

Assign color 1 to vertex V_n. All edges are incident at V_n. So assign different colors to each edge. Assign color 2 to the first edge e_1 = {V_1, V_n}, color 3 to e_2 = {V_2, V_n},..., color n to an edge e_{n-1} = {V_{n-1}, V_n}. Consequently assign colors 3, 4, ... n-1, 1, 2 to vertices V_1, V_2, . . . V_{n-1} respectively.

Thus SPC (G) = n = Δ (G) + 1, where Δ (G) = n-1.

As graph G has only one region so assign different color to that region. Therefore PC (G) = n + 1 = Δ (G) + 2. □

Theorem 2.5 *If G is a tree graph on n ≥ 3 vertices then SPC (G) = Δ (G) + 1 and PC (G) = Δ (G) + 2, where Δ (G) = Highest degree of a vertex in G.*

Proof: Let G be a tree on n ≥ 3 vertices. Find the largest subgraph H of a tree which is a star graph. By above theorem 5.27, SPC (H) = Δ (H) + 1 and PC (H) = Δ (H) + 2. Therefore same numbers of colors are sufficient for coloring of graph G. Thus, SPC (G) = Δ (G) + 1 and PC (G) = Δ (G) + 2. □

Theorem 2.6 *If C_n is a cycle graph on n ≥ 3 vertices then SPC (C_{3n}) = 3, SPC (C_{3n+1}) = 4, SPC (C_{3n+2}) = 4 and PC (C_{3n}) = 5, PC (C_{3n+1}) = 6, PC (C_{3n+2}) = 6.*

Proof: Without loss of generality, suppose G, H and K are graphs on 6, 5 and 7 vertices as given in figure 5.12.

In a graph G, we have six vertices as A, B, C, D, E, F and edges e_1, e_2, e_3, e_4, e_5, e_6 as shown in figure 5.12 Consider the closed path A-e_1-B- e_2 -C-e_3-D- e_4 –E- e_5-F- e_6-A. Now, assign colors successively 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, 1 to every element of the path. Therefore SPC (G) = 3.

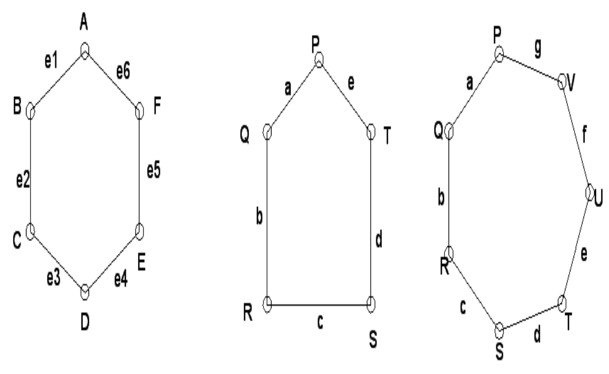

Figure 2.4: Cycle Graphs

There are two regions in G and all vertices and edges lie on the boundary of those regions. So assign two different colors to these regions. Thus PC (G) = 3 + 2 = 5.

In a graph H, we have five vertices say P, Q, R, S, T and five edges say a, b, c, d, e. Consider the closed path P-a-Q-b- R-c- S-d-T-e-P. Now assign a different color to each element of the path. We assign colors 1, 2, 3, 1, 2, 3, 1, 2, 3 respectively to P-a- Q-b- R-c- S-d-T. An edge e = {T, P} is adjacent to edge d.

So we have to assign color different from 1, 2, 3 to edge e. Consequently assign color 4 to edge e. Thus SPC (H) = 4 and PC (G) = 4 + 2 = 6.

In a graph K, the closed path is P-a- Q-b- R-c- S-d-T-e-U-f-V-g-P. We assign colors 1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 4, 3, 2, 4, 1 respectively to P – a - Q – b – R – c - S- d -T- e -U- f - V- g -P. Thus SPC (H) = 4 and PC (G) = 4 + 2 = 6.

Hence, in general SPC (C_{3n}) = 3, SPC (C_{3n+1}) = 4, SPC (C_{3n+2}) = 4 and PC (C_{3n}) = 5, PC (C_{3n+1}) = 6 and PC (C_{3n+2}) = 6. □

Theorem 2.7 *If G is a rose graph with m ≥ 2 loops then SPC (G) = m + 1 and PC (G) = m + 2.*

Proof: Without loss of generality, assume that H is a rose graph with five loops as given below.

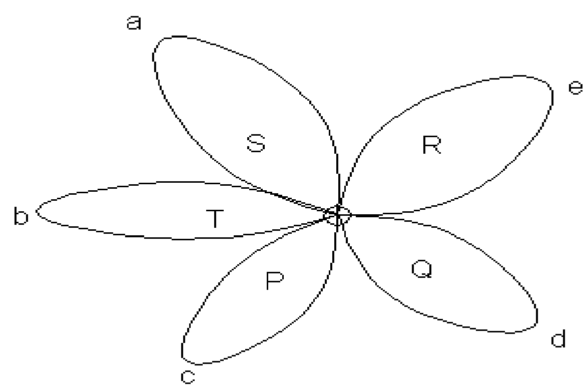

Figure 2.5: Rose Graph

Assign color 1 to vertex V. Five different edges are incident at V, so assign five different colors to edges a, b, c, d and e. Now assign color of edge a to region P, color of edge b to region Q, color of edge c to region R, color of edge d to region S, color of edge e to region T. The vertex V and these five loops are the boundaries of an infinite region. So assign a different color to this infinite region. Thus we required seven different colors for the perfect coloring of H. Therefore SPC (H) = 5 + 1 = 6 and PC (H) = 5 + 2 = 7.

In general, if G is a rose graph with m loops then assign m colors to these m loops. We use same colors for regions, one different color for vertex and one more different color for an infinite region. Thus SPC (G) = m + 1 and PC (G) = m + 2. □

III. ACKNOWLEDGMENT

Thanks to Dr. J. N. Salunke for giving me strength and ability to do independent research in graph theory and computer science. I also thanks to authorities of this national conference and expert who have contributed for the same.

REFERENCES

[1] Appel K. and W. Haken, Every Planar Map is Four Colorable, Bulletin of American Mathematical society, 82 (1977), 711-712.

[2] Appel K. and W. Haken, Every Planar Map is Four Colorable, Contemporary Mathematics 98, American Mathematical society, 1989.

[3] Brooks R.L., On Coloring of Nodes of a Network, Proc. Cambridge Phil. Society, Vol. 37, 1941, 194-197.

[4] H. R. Bhapkar and J. N. Salunke, *isomorphism of graphs, in International Journal of Mathematical Sciences and Engineering Applications, Vol. 8, No. II, 0973-9424, March 14.

[5] H. R. Bhapkar and J. N. Salunke, The Geometric Dual of HB Graph, *outerplanar Graph and Related Aspects, in Bulletin of Mathematical Sciences and Applications, ISSN 2278-9634, Volume 3, No. 3, pp 114-119, August 2014.

[6] H. R. Bhapkar and J. N. Salunke, "Proof of Four Color Map Theorem by Using PRN of Graph", in online journal of Bulletin of Society for Mathematical Services and Standards ISSN 2277-8020, Volume 3, No. 2, pp 35-42, Sept. 2014.

[7] Narsingh Deo, Graph Theory with Applications To Engineering and Computer Science, Prentice –Hall of India,2003,88-111.

[8] Robin J. Wilson, Introduction to Graph Theory, Pearson, 978-81-317-0698-5, 2011

[9] Tait P.G., On the coloring of maps, Proceeding Royal Society, Edinburgh Sect. A 10 (1878-1810), 501-503,729

[10] V. K. Balakrishnan, Schaum's outline of theory and problems of graph theory, 198-243, 2008.