Boundary Conformal Volume And First Eigenvalue

Habeeb. I.A. Ibrahim[1], Mohammed Nour A.Rabih[2], Osman Abdalla Adam Osman[3](Email: o.osman@qu.edu.sa)
[1, 2, 3] Dept of Mathematics
[1]Feculty of Education University of Zaligei –Sudan
[1]College of Science & Arts in Elmiznab – Qassim University – Saudia Arabia.
[2]College of Science - University of Bakht Er-ruda-Eddwaim –Sudan
[2, 3]College of Science & Arts in Uglat Asugour – Qassim University – Saudia Arabia.

Abstract- *In this paper we give an overview of results about the boundary and relative conformal volume to manifolds* Σ *, and we prove that* $Vol_{bc}(\Sigma, n, \varphi) \le 2\pi$ *and* $Vol_{bc}(\Sigma, n, \varphi) \le L^2/2A$ *, where* $Vol_{bc}(\Sigma, n, \varphi)$ *is a boundary n-conformal volume of* φ *, we also prove that* $Vol_{rc}(\Sigma, n) \ge Vol_{rc}(\Sigma, n+\epsilon)$

Keywords- Conformal, Volume, Eigenvalue , Riemannian manifold , Laplacian.

I. INTRODUCTION

Let (Σ^k, g) be a k-dimensional compact Riemannian manifold with boundary $\partial\Sigma \neq \emptyset$, and let B^n be the unit ball in $\mathbb{R}^n$. Assume that Σ admits a conformal map $\varphi: \Sigma \to B^n$ with $\varphi(\partial\Sigma) \subset \partial B^n$. Let G be the group of conformal diffeomorphisms of B^n. We define the boundary conformal volume to be the Li-Yau [2] conformal volume of the boundary submanifold $\partial\Sigma$. we give estimates for the first eigenvalue of the Dirichlet-to-Neumann map which are analogs of the estimates of [2] and [5] for the first Neumann eigenvalue of the Laplacian[1].

Definition .1 : Given a map $\varphi \in C^1(\partial\Sigma, \partial B^n)$ that admits a conformal extension $\varphi: \Sigma \to B^n$, define the boundary n-conformal volume of φ by:

$$\mathrm{Vol}_{bc}(\Sigma, n, \varphi) = \sup_{f \in G} \mathrm{Vol}\left(f\left(\varphi(\partial\Sigma)\right)\right).$$

The boundary n-conformal volume of Σ is then defined to be:

$$\mathrm{Vol}_{bc}(\Sigma, n) = \inf_{\varphi} \mathrm{Vol}_{bc}(\Sigma, n, \varphi).$$

where the infimum is over all $\varphi \in C^1(\partial\Sigma, \partial B^n)$ that admit conformal extensions $\varphi: \Sigma \to B^n$. It can be shown (see Lemma 7) that $\mathrm{Vol}_{bc}(\Sigma, n) \ge \mathrm{Vol}_{bc}(\Sigma, n+1)$. The boundary conformal volume of Σ is defined to be:

$$\mathrm{Vol}_{bc}(\Sigma) = \lim_{n\to\infty} \mathrm{Vol}_{bc}(\Sigma, n).$$

Note that: For any k-dimensional manifold Σ with boundary, the boundary n-conformal volume of Σ is bounded below by the volume of the $(k-1)$-dimensional sphere:

$$\mathrm{Vol}_{bc}(\Sigma, n) \ge \mathrm{Vol}(\mathbb{S}^{k-1}).$$

The proof is as in [2]; given a point θ on $\mathbb{S}^{n-1}$, let $f_\theta(t)$ be the one parameter subgroup of the group of conformal diffeomorphisms of the sphere generated by the gradient of the linear functions of $\mathbb{R}^n$ in the direction θ. For all $t, f_\theta(t)$fixes the points θ and $-\theta$, and$\lim_{t\to\infty} f_\theta(t)(x) = \theta$ for all $x \in \mathbb{S}^{n-1} \setminus \{-\theta\}$. If $\varphi: \partial\Sigma \to \mathbb{S}^{n-1}$is a map whose differential has rank $k-1$ at x, then,

$$\lim_{t\to\infty} \mathrm{Vol}\left(f_{-\varphi(x)}(t)\left(\varphi(\partial\Sigma)\right)\right) = m\mathrm{Vol}(\mathbb{S}^{k-1})$$

for some $m \in \mathbb{Z}^+$ (here the integer m is the multiplicity of the immersed submanifold $\partial\Sigma$ atthe point $-\theta$).

For $k = 2$ and for a minimal surface Σ that is a solution to the free boundary problem inthe unit ball B^n in $\mathbb{R}^n$, the boundary n-conformal volume of Σ is the length of the boundaryof Σ; that is, its boundary length is maximal in its conformal orbit.

Theorem .2 : Let Σ a minimal surface [1] in B^n, with nonempty boundary $\partial\Sigma \subset \partial B^n$, and meeting ∂B^n orthogonally along $\partial\Sigma$, given by the isometric immersion $\varphi: \Sigma \to B^n$. Then, $\mathrm{Vol}_{bc}(\Sigma, n, \varphi) = L(\partial\Sigma)$,

The length of the boundary of Σ.

Proof:

The trace-free second fundamental form $\left\|A - \frac{1}{2}(Tr_g A)g\right\|^2 dV_g$ is conformally invariant for surfaces.

Using the Gauss equation, we have $2\left\|A-\frac{1}{2}(Tr_g A)g\right\|^2 = H^2-4K$. Therefore, given any $f \in G$, $\int_\Sigma (H^2-4K)\,da = \int_{f(\Sigma)} (\tilde{H}^2-4\tilde{K})\,d\tilde{a},$

Where $d\tilde{a}$ denotes the induced area element on $f(\Sigma)$, and $\tilde{K}$ and $\tilde{H}$ denote the Gauss and mean curvatures of $f(\Sigma)$ in $\mathbb{R}^n$. Since Σ is minimal, $H=0$, and so we have,

$$-4\int_\Sigma K\,da = \int_{f(\Sigma)} \tilde{H}^2\,d\tilde{a} - 4\int_{f(\Sigma)} \tilde{K}\,d\tilde{a}. (1)$$

By the Gauss-Bonnet Theorem,

$$\int_\Sigma K\,da = 2\pi\chi(\Sigma) - \int_{\partial\Sigma} k\,ds$$
$$\int_{f(\Sigma)} \tilde{K}\,da = 2\pi\chi(f(\Sigma)) - \int_{\partial f(\Sigma)} \tilde{k}\,ds,$$

and using this in (1), since $\chi(\Sigma) = \chi(f(\Sigma))$, we obtain

$$4\int_{\partial\Sigma} k\,ds = \int_{f(\Sigma)} \tilde{H}^2\,d\tilde{a} + 4\int_{\partial f(\Sigma)} \tilde{k}\,d\tilde{s}\,(2)$$
$$\geq 4\int_{\partial f(\Sigma)} \tilde{k}\,d\tilde{s}$$

If T is the oriented unit tangent vector of $\partial\Sigma$ and ν is the inward unit conormal vectoralong $\partial\Sigma$, then.

$$k = \langle \frac{dT}{ds}, \nu\rangle = -\langle T, \frac{d\nu}{ds}\rangle = \langle T, \frac{d\varphi}{ds}\rangle = \langle T, T\rangle = 1,$$

where in the third to last equality we have used the fact that $\nu = -\varphi$ since Σ meets ∂B^n orthogonally along $\partial\Sigma$ Since f is conformal, $f(\Sigma)$ also meets ∂B^n orthogonally along $\partial f(\Sigma)$,and so we also have that $\tilde{k}=1$. Using this in (2) we obtain.

$$L(\partial\Sigma) \geq L(\partial f(\Sigma)).$$

This shows that

$$L(\partial\Sigma) \geq \mathrm{Vol}_{bc}(\Sigma, n, \varphi)$$

as claimed.

The proof of Theorem **2** implies that any minimal surface [1], that is a solution to the free boundary problem in the unit ball in $\mathbb{R}^n$ has area greater than or equal to that of a flat equatorial disk solution.

Theorem .3 : Let Σ be a minimal surface in B^n, with (nonempty) boundary $\partial\Sigma \subset \partial B^n$, and meeting ∂B^n orthogonally along $\partial\Sigma$. Then,

$$2A(\Sigma) = L(\partial\Sigma) \geq 2\pi.$$

Proof: Given $f \in G$, as in the proof Theorem **4**, we have,

$$L(\partial\Sigma) \geq L(\partial f(\Sigma)). \qquad (3)$$

Since Σ is minimal, the coordinate functions are harmonic $\Delta_\Sigma x^i = 0$, and $\Delta_\Sigma |x|^2 = 4$. Therefore,

$$4A(\Sigma) = \int_\Sigma \Delta_\Sigma |x|^2\,da = \int_{\partial\Sigma} \frac{\partial |x|^2}{\partial\nu} ds = \int_{\partial\Sigma} 2\,ds = 2L(\partial\Sigma).$$

Using this in (3) gives,

$$2A(\Sigma) \geq L(\partial f(\Sigma)).$$

If $p \in \partial\Sigma$, then,

$$\lim_{t\to\infty} L\left(f_p(t)(\partial\Sigma)\right) = mL(\mathbb{S}^1) = 2\pi m$$

For some $m \in \mathbb{Z}^+$, and so, we have the desired conclusion. $2A(\Sigma) = L(\partial\Sigma) \geq 2\pi.$

Corollary .4 : The sharp isoperimetric inequality [3], holds for free boundary minimal surfacesin the ball:

$$A \leq \frac{L^2}{4\pi}.$$

Proof: For free boundary minimal surfaces in the ball we have $2A(\Sigma) = L(\partial\Sigma)$, as shown in the proof of Theorem **3**. It follows that the inequality $A(\Sigma) \geq \pi$ is equivalent to the sharp isoperimetric inequality $A \leq L^2/4\pi$.

Corollary .5 : Show that

(i) $\mathrm{Vol}_{bc}(\Sigma, n, \varphi) \leq 2\pi$
(ii) $\mathrm{Vol}_{bc}(\Sigma, n, \varphi) \leq L^2/2A$

Proof :(i) Theorem **2** and Theorem **3** show that $\mathrm{Vol}_{bc}(\Sigma, n, \varphi) \leq 2\pi$

(ii) Since $A \leq \frac{L^2}{4\pi}$ then,

$$\text{Vol}_{bc}(\Sigma, n, \varphi) \leq 2\pi \leq \frac{L^2}{4\pi}$$

Definition .6 : Let Σ be a k-dimensional compact Riemannian manifold [5], with boundary that admits a conformal map $\varphi: \Sigma \to B^n$ with $\varphi(\partial\Sigma) \subset \partial B^n$. Define the relative n-conformal volume of φ by.

$$\text{Vol}_{rc}(\Sigma, n, \varphi) = \sup_{f \in G} \text{Vol}\left(\left(f(\varphi(\Sigma))\right)\right).$$

The relative n-conformal volume of Σ is then defined to be:

$$\text{Vol}_{rc}(\Sigma, n) = \inf_{\varphi} \text{Vol}_{rc}(\Sigma, n, \varphi)$$

Where the infimum is over all non-degenerate conformal maps $\varphi: \Sigma \to B^n$ with $\varphi(\partial\Sigma) \subset \partial B^2$.

Lemma .7 : If $m \geq n$, then $\text{Vol}_{rc}(\Sigma, n) \geq \text{Vol}_{rc}(\Sigma, m)$.

Proof: To see this, suppose $\varphi: \Sigma \to B^n \subset B^m$ is conformal, with $\varphi(\partial\Sigma) \subset \partial B^n \subset \partial B^m$. Let $A = \varphi(\Sigma) \subset B^n$ and suppose that f is a conformal transformation of B^m. Then $f(A)$ lies in the spherical cap $f(B^n)$ in B^m whose boundary lies in ∂B^m. Let $T \in O(m)$ be an orthogonal transformation that rotates this spherical cap so that its boundary lies in an n-plane parallel to the n-plane containing the boundary of the original equatorial B^n. Let P be the conformal projection of $T\left(f(B^n)\right)$ onto B^n, and let $A' = P\left(T\left(f(A)\right)\right)$. Clearly P is volume increasing, and so.

$$\text{Vol}(A') \geq \text{Vol}\left(f(A)\right)$$

But A' is the image of A under some conformal transformation of B^n, therefore,

$$\sup_{F \in G} \text{Vol}\left(F(A)\right) \geq \sup_{f \in G'} \text{Vol}\left(f(A)\right)$$

Where G denotes the group of conformal transformations of B^n, and G' denotes the group of conformal transformations of B^m.

The relative conformal volume of Σ is defined to be,

$$\text{Vol}_{rc}(\Sigma) = \lim_{n \to \infty} \text{Vol}_{rc}(\Sigma, n)$$

Note that : For any k-dimensional manifold Σ with boundary, the relative n-conformal volume of Σ is bounded below by the volume of the k-dimensional ball:

$$\text{Vol}_{rc}(\Sigma, n) \geq \text{Vol}(B^k).$$

To see this, suppose $\varphi: \Sigma \to B^n$ is a conformal map with $\varphi(\partial\Sigma) \subset \partial B^n$, whose differential has rank k at $x \in \partial\Sigma$. The conformal diffeomorphisms $f_{-\varphi(x)}(t)$ of the sphere, extend to conformal diffeomorphisms of B^n, and,

$$\lim_{t \to \infty} \text{Vol}\left(f_{-\varphi(x)}(t)\left(\varphi(\Sigma)\right)\right) = m\text{Vol}(B^k)$$

For some $m \in \mathbb{Z}^+$, the multiplicity of $\varphi(\partial\Sigma)$ at $\varphi(x)$.

Corollary .8 : Show that $\text{Vol}_{rc}(\Sigma, n) \geq \text{Vol}_{rc}(\Sigma, n+\epsilon)$.

Proof:

For $\epsilon > 0$ suppose $\varphi: \Sigma \to B_j^n \subset B_j^{n+\epsilon}$ is conformal, with $\varphi(\partial\Sigma) \subset B_j^n \subset \partial B_j^{n+\epsilon}$. Let $A = \varphi(\Sigma) \subset B_j^n$ and suppose that f is a conformal transformation of $B_j^{n+\epsilon}$. Then $\sum_{j=1}^r f(A_j)$ lies in the spherical cap $\sum_{j=1}^r f(B_j^n)$in $B_j^{n+\epsilon}$. whose boundary lies in $\partial B_j^{n+\epsilon}$. Let $T \in O(m)$ be an orthogonal transformation that rotates this spherical cap so that its boundary lies in an n-plane parallel to the n-plane containing the boundary of the original equatorial B_j^n. Let P be the conformal projection of $T\left(\sum_{j=1}^r f(B_j^n)\right)$onto B_j^n, and let $\sum_{j=1}^r A_j' = P\left(T\left(\sum_{j=1}^r f(A_j)\right)\right)$. Clearly P is volume increasing, and so,

$$\text{Vol}\left(\sum_{j=1}^r A_j'\right) \geq \text{Vol}\left(\sum_{j=1}^r f(A_j)\right)$$

But A_j' is the image of A_j under some conformal transformation of B_j^n, Hence,

$$\sup_{F \in G} \text{Vol}\left(\sum_{j=1}^r f(A_j)\right) \geq \sup_{f \in G'} \text{Vol}\left(\sum_{j=1}^r f(A_j)\right)$$

Where G is the group of conformal transformations of B_j^n, and G' denotes the group of conformal transformations of $B_j^{n+\epsilon}$.

The relative conformal volume of Σ is defined to be.

$$\mathrm{Vol}_{rc}(\Sigma) = \lim_{n\to\infty} \mathrm{Vol}_{rc}(\Sigma, n)$$

Lemma .9 : Let (M, g) be a compact Riemannian manifold, and let φ be an immersion of M into $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. There exists $f \in G$ such that $\psi = fo\varphi = (\psi^1, \dots \psi^n)$ satisfies:

$$\int_M \psi^i \, dv_g = 0$$

For $i = 1, \dots, n$.

Theorem .10 : Let (Σ, g) be a compact k-dimensional Riemannian manifold [5], with nonempty boundary. Let $\sigma_1 > 0$ be the first non-zero eigenvalue of the Dirichlet-to-Neumann map [1], on (Σ, g). Then,

$$\sigma_1 \mathrm{Vol}(\partial\Sigma)\mathrm{Vol}(\Sigma)^{\frac{2-k}{k}} \le k \, \mathrm{Vol}_{rc}(\Sigma, n)^{\frac{2}{k}}$$

For all n for which $\mathrm{Vol}_{rc}(\Sigma, n)$ is defined (i.e. such that there exists a conformal mapping $\varphi: \Sigma \to B^n$ with $\varphi(\partial\Sigma) \subset \partial B^n$). Equality implies that there exists a conformal harmonic map $\varphi: \Sigma \to B^n$ which (after rescaling the metric g) is an isometry on $\partial\Sigma$, with $\varphi(\partial\Sigma) \subset \partial B^n$ and such that $\varphi(\Sigma)$ meets ∂B^n orthogonally along $\varphi(\partial\Sigma)$. For $k > 2$ this map is an isometric minimal immersion of Σ to its image. Moreover, the immersion is given by a subspace of the first eigenspace.
The following is an immediate consequence of the theorem.

Corollary .11 : Let Σ be a compact surface with nonempty boundary and metric g. Let $\sigma_1 > 0$ be the first non-zero eigenvalue of the Dirichlet-to-Neumann map on (Σ, g). Then

$$\sigma_1 L(\partial\Sigma) \le 2 \, \mathrm{Vol}_{rc}(\Sigma, n)$$

for all n for which $\mathrm{Vol}_{rc}(\Sigma, n)$ is defined. Equality implies that there exists a conformal minimal immersion $\varphi: \Sigma \to B^n$ by first eigenfunctions which (after rescaling the metric) is anisometry on $\partial\Sigma$, with $\varphi(\partial\Sigma) \subset \partial B^n$ and such that $\varphi(\Sigma)$ meets ∂B^n orthogonally along $\varphi(\partial\Sigma)$.

Proof. Let $\varphi: \Sigma \to B^n$ be a conformal map with $\varphi(\partial\Sigma) \subset \partial B^n$.

By Lemma **9** we can assume that $\varphi = (\varphi^1, \dots, \varphi^n)$ satisfies:

$$\int_{\partial\Sigma} \varphi^i \, ds = 0$$

for $i = 1, \dots, n$. Let $\hat{\varphi}^i$ be a harmonic extension of $\varphi^i|_{\partial\Sigma}$.

Then,

$$\sigma_1 \le \frac{\int_\Sigma |\nabla \hat{\varphi}^i|^2 dv_\Sigma}{\int_{\partial\Sigma} (\varphi^i)^2 dv_{\partial\Sigma}} \le \frac{\int_\Sigma |\nabla \varphi^i|^2 dv_\Sigma}{\int_{\partial\Sigma} (\varphi^i)^2 dv_{\partial\Sigma}}. \qquad (4)$$

By Holder's inequality, and since φ is conformal then,

$$\int_\Sigma \sum_{i=1}^n |\nabla \varphi^i|^2 dv_\Sigma \le \mathrm{Vol}(\Sigma)^{\frac{k-2}{k}} \left[\int_\Sigma \left(|\nabla \varphi^i|^2 \right)^{\frac{k}{2}} dv_\Sigma \right]^{\frac{2}{k}} =$$

$$\mathrm{Vol}(\Sigma)^{\frac{k-2}{k}} \left[k^{\frac{k}{2}} \mathrm{Vol}(\varphi(\mathrm{T})) \right]^{\frac{2}{k}} \le$$

$$k\mathrm{Vol}(\Sigma)^{\frac{k-2}{k}} \mathrm{Vol}_{rc}(\Sigma, n, \varphi)^{\frac{2}{k}}.$$

On the other hand, since $\varphi(\partial\Sigma) \subset \partial B^n$,

$$\sum_{i=1}^n \int_{\partial\Sigma} (\varphi^i)^2 dv_{\partial\Sigma} = \int_{\partial\Sigma} dv_{\partial\Sigma} = \mathrm{Vol}(\partial\Sigma).$$

Then by (4) we have,

$$\sigma_1 \mathrm{Vol}(\partial\Sigma)\mathrm{Vol}(\Sigma)^{\frac{2-k}{k}} \le k\mathrm{Vol}_{rc}(\Sigma, n, \varphi)^{\frac{2}{k}}.$$

Since $\mathrm{Vol}_{rc}(\Sigma, n) = \inf_\varphi \mathrm{Vol}_{rc}(\Sigma, n, \varphi)$ we get.

$$\sigma_1 \mathrm{Vol}(\partial\Sigma)\mathrm{Vol}(\Sigma)^{\frac{2-k}{k}} \le k \, \mathrm{Vol}_{rc}(\Sigma, n)^{\frac{2}{k}}.$$

Now assume that we have equality, $\sigma_1 \mathrm{Vol}(\partial\Sigma) = kV_{rc}(\Sigma, n)^{2/k} V(\Sigma)^{(k-2)/k}$. Choose a sequence of conformal maps $\varphi: \Sigma \to B^n$ with $\varphi_j(\partial\Sigma) \subset \partial B^n$, such that,

$$\lim_{j\to\infty} \mathrm{Vol}_{rc}(\Sigma, n, \varphi_j) = \mathrm{Vol}_{rc}(\Sigma, n)$$

and by composing with a conformal transformation of the ball we may assume:

$$\int_{\partial\Sigma} \varphi_j^i \, ds = 0$$

for all i, j. By changing the order of coordinates, we may assume that:

$$\lim_{j\to\infty}\int_{\Sigma}\left(\varphi_j^i\right)^2 da \begin{cases} > 0 & i = 1, \dots, N \\ = 0 & i = N+1, \dots n \end{cases}$$

We have:

$$\sigma_1 \mathrm{Vol}(\partial\Sigma) = \sigma_1 \sum_{i=1}^{n}\int_{\partial\Sigma}\left(\varphi_j^i\right)^2 dv_{\partial\Sigma} \leq \sum_{i=1}^{n}\int_{\Sigma}\left|\nabla\varphi_j^i\right|^2 dv_{\Sigma} \leq \mathrm{Vol}(\Sigma)^{\frac{k-2}{k}}\left[\int_{\Sigma}\left(\sum_{i=1}^{n}\left|\nabla\varphi_j^i\right|^2\right)^{\frac{k}{2}} dv_{\Sigma}\right]^{\frac{2}{k}} \leq k\mathrm{Vol}_{rc}\left(\Sigma, n, \varphi_j\right)^{\frac{2}{k}}\mathrm{Vol}(\Sigma)^{\frac{k-2}{k}}$$

Letting $j \to \infty$ and using $\sigma_1 \mathrm{Vol}(\partial\Sigma) = k\mathrm{Vol}_{rc}(\Sigma, n)^{2/k}\mathrm{Vol}(\Sigma)^{(k-2)/k}$ we get:

$$\sigma_1 \mathrm{Vol}(\partial\Sigma) = \sigma_1 \lim_{j\to\infty}\sum_{i=1}^{n}\int_{\partial\Sigma}\left(\varphi_j^i\right)^2 dv_{\partial\Sigma} = \lim_{j\to\infty}\sum_{i=1}^{n}\int_{\Sigma}\left|\nabla\varphi_j^i\right|^2 dv_{\Sigma} = \mathrm{Vol}(\Sigma)^{\frac{k-2}{k}}\lim_{j\to\infty}\left[\int_{\Sigma}\left(\sum_{i=1}^{n}\left|\nabla\varphi_j^i\right|^2\right)^{\frac{k}{2}} dv_{\Sigma}\right]^{\frac{2}{k}} = \sigma_1 \mathrm{Vol}(\partial\Sigma) \quad (5)$$

Therefore, for any fixed $i, \{\varphi_j^i\}$ us a bounded sequence in $W^{1,k}(\Sigma, \mathbb{R})$, and since the inclusion $W^{1,k}(\Sigma, \mathbb{R}) \subset L^2(\Sigma, \mathbb{R})$ is compact, by passing to a subsequence we can assume that $\{\varphi_j^i\}$ converges weakly in $W^{1,k}(\Sigma, \mathbb{R})$, strongly in $L^2(\Sigma, \mathbb{R})$, and point wise a.e., to map $\psi^i: \Sigma \to \mathbb{R}$. Clearly $\sum_{i=1}^{n}(\psi^i)^2 \leq 1$ a.e. on $\Sigma, \sum_{i=1}^{n}(\psi^i)^2 = 1$ a.e. on $\partial\Sigma$, and $\psi^i = 0$ for $i = N+1, \dots n$. Since for all i.

$$\sigma_1 \int_{\partial\Sigma}\left(\varphi_j^i\right)^2 dv_{\partial\Sigma} \leq \int_{\Sigma}\left|\nabla\varphi_j^i\right|^2 dv_{\Sigma}.$$

And

$$\sigma_1 \lim_{j\to\infty}\sum_{i=1}^{n}\int_{\partial\Sigma}\left(\varphi_j^i\right)^2 dv_{\partial\Sigma} = \lim_{j\to\infty}\sum_{i=1}^{n}\int_{\Sigma}\left|\nabla\varphi_j^i\right|^2 dv_{\Sigma},$$

We have:

$$\lim_{j\to\infty}\int_{\Sigma}\left|\nabla\varphi_j^i\right|^2 dv_{\Sigma} = \sigma_1 \lim_{j\to\infty}\int_{\partial\Sigma}\left(\varphi_j^i\right)^2 dv_{\partial\Sigma} =$$
$$\sigma_1 \int_{\partial\Sigma}(\psi^i)^2 dv_{\partial\Sigma} \leq \int_{\Sigma}\left|\nabla\varphi^i\right|^2 dv_{\Sigma}. \quad (6)$$

On the other hand, $\varphi_j^i \to \psi^i$ weakly in $W^{1,k}(\Sigma, \mathbb{R})$, and so,

$$\int_{\Sigma}\left|\nabla\psi^i\right|^2 dv_{\Sigma} \leq \lim_{j\to\infty}\int_{\Sigma}\left|\nabla\varphi_j^i\right|^2 dv_{\Sigma}$$

Therefore, we must have equality in (6), and so,

$$\lim_{j\to\infty}\int_{\Sigma}\left|\nabla\varphi_j^i\right|^2 dv_{\Sigma} = \int_{\Sigma}\left|\nabla\psi^i\right|^2 dv_{\Sigma}$$

which means $\{\varphi_j^i\}$converges to ψstrongly in $W^{1,2}(\Sigma, \mathbb{R})$. Moreover,

$$\sigma_1 \int_{\partial\Sigma}(\psi^i)^2 dv_{\partial\Sigma} = \int_{\Sigma}\left|\nabla\psi^i\right|^2 dv_{\Sigma}$$

and it follows that $\{\psi_i\}_{i=1}^{N}$ are first eigenfunctions. In particular, $\psi^i$is harmonic for $i = 1, \dots, N$. Also, since $\varphi_j$is conformal and converges strongly in $W^{1,2}$ to ψ, the map:

$$\psi: \Sigma \to B^N$$
$$x \mapsto (\psi^1(x), \dots, \psi^N(x))$$

defines a conformal map. Therefore, $\psi: \Sigma \to B^N$is conformal and harmonic, with $\psi(\partial\Sigma) \subset \partial B^N$. Since $\psi(\partial\Sigma) \subset \partial B^N$and

$$\frac{\partial\psi}{\partial v} = \sigma_1 \psi \quad (7)$$

on $\partial\Sigma$ since $\psi^i$are eigenfunctions, it follows that $\psi(\Sigma)$ meets ∂B^Northogonally along $\psi(\partial\Sigma)$.

By scaling the metric we can assume that $\sigma_1 = 1$. Then by (7), on $\partial\Sigma$ we have:

$$\left|\frac{\partial\psi}{\partial v}\right| = |\psi| = 1,$$

and hence $\psi$is an isometry on $\partial\Sigma$. Finally, for $k > 2$ we have from (5)

$$\lim_{j\to\infty}\sum_{i=1}^{n}\int_{\Sigma}\left|\nabla\varphi_j^i\right|^2 dv_{\Sigma} = \sum_{i=1}^{n}\int_{\Sigma}\left|\nabla\psi^i\right|^2 dv_{\Sigma} =$$
$$\mathrm{Vol}(\Sigma)^{\frac{k-2}{k}}\lim_{j\to\infty}\left[\int_{\Sigma}\left(\sum_{i=1}^{n}\left|\nabla\varphi_j^i\right|^2\right)^{\frac{k}{2}} dv_{\Sigma}\right]^{\frac{2}{k}}$$

By lower semicontinuity of the norm under weak convergence this implies

$$\int_{\Sigma} |\nabla \psi|^2 dv_{\Sigma} = \mathrm{Vol}(\Sigma)^{\frac{k-2}{k}} \left[\int_{\Sigma} \left(\sum_{i=1}^{n} |\nabla \psi^i|^2 \right)^{\frac{k}{2}} dv_{\Sigma} \right]^{\frac{2}{k}}$$

Now the Holder inequality implies the opposite inequality and thus we have equality in the Holder inequality, which implies $|\nabla\psi|^2$ is constant on Σ, and this constant must be k by the boundary normalization. Since ψ is conformal this implies that $\psi$is an isometry as claimed.

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