List Circular Coloring Of Trees And Cycles

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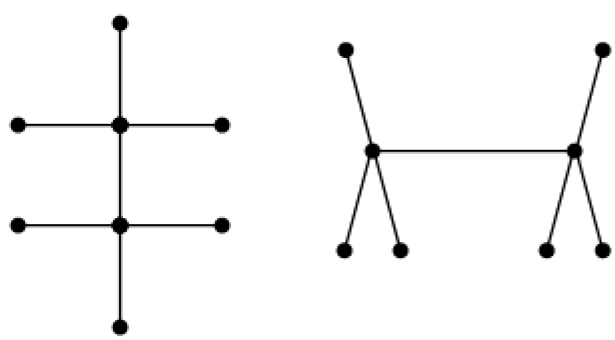
I. INTRODUCTION

In this section we deal with a particular type of connected graphs called trees. These graphs are important for their applications in different fields. The concept of a tree was introduced by Cayley in 1857. Tree is the simplest graph which is convenient to study and to prove any result on graph theory.

Definition:

A graph that does not contain any cycle is called an acyclic graph.

A connected acyclic graph is called a tree. Trees with 8 vertices are given in fig.



Note Union of trees is called forest.

Theorem:

Let *G* be a graph. The following statements are equivalent

i) *G* is a tree.
ii) Every two vertices of *G* are joined by a unique path
iii) *G* is connected and $q = p - 1$.
iv) *G* is acyclic and $q = p - 1$
.

II. SOME PROPERTIES OF TREES

Theorem: If in a graph *G* there is one and only one path between every pair of vertices, *G* is a tree.

Proof:

Existence of a path between every pair of vertices assume that *G* is connected.

A circuit in a graph (with two or more vertices) implies that there is at least one pair of vertices a, b such that there are two distinct paths between *a* and *b*.

Since *G* has one and only one path between every pair of vertices, *G* can have no circuits. Therefore *G* is a tree. Hence the proof.

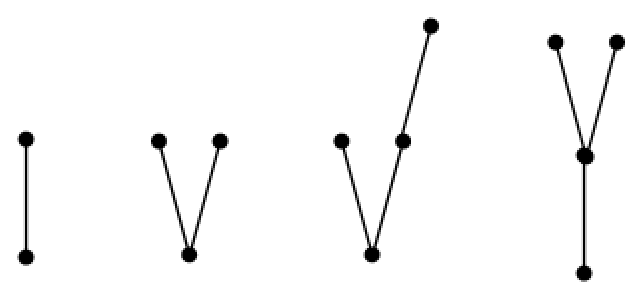
Theorem:

A tree with *n* vertices has $n-1$ edges.

Proof:

The theorem will be proved by induction on the number of vertices.

It is easy to see that the theorem is true for $n = 1, 2, \text{ and } 3$ in the figure.



Tree with one, two, three and four vertices

Assume that the theorem holds for all trees with fewer than *n* vertices.

Let us now consider a tree *T* with *n* vertices. In *T* let e_k be an edge with end vertices v_1 and v_2.

According to theorem 5.2.1 there is no other path between v_i and v_j except e_k.

Therefore, deletion of e_k from *T* will disconnect the graph.

Furthermore, $T-e_k$ consists of exactly two components, and since there were no circuits in *T* to begin with, each of these components is a tree.

Both these trees, t_1 and t_2 have fewer than *n* vertices each and therefore, by the induction hypothesis, each contain one less edge than the number of vertices in it.

Thus $T-e_k$ consists of $n-2$ edges (and *n* vertices). Hence *T* has exactly $n-1$ edges.

Hence the proof.

III. COLOROING THE TREES

First we introduce some notation that will used. Suppose *p* is a positive integer. Then for any integer *t*, $[t]_p$ denotes the remainder of *t* upon the division by *p*, that is $[t]_p$ is the unique integer $0 \le t' < p$ such that $t-t'$ is a multiple of *p*. In (p,q)- colorings of graphs, the color set is $Z_p=\{0,1,\cdots,p-1\}$.

The summation in colors are all modulo *p*, and any integer *t* for which $[t]_p=i$ can be used to represent the color *i*.

For example, When we say “color a vertex *x* with color 2*p*” it means to color *x* with color 0.

Moreover, the colors are viewed to form a circle, that is, the integers $0,1,\cdots,p-1$ are cyclically ordered.

If $a,b\in\{0,1,\cdots p-1\}$, then $[a,b]_p$ denotes the set of cyclically consecutive elements of the set $\{0,1,\cdots,p-1\}$ from *a* to *b*. That is, $[a,b]_p=\{t:[t-a]_p\le[b-a]_p\}$

For example, $[2,5]_p=[2,3,4,5]$ and $[5,2]_p=\{5,6,\cdots,p-1,0,1,2\}$.

v_j t_2

The set $[a,b]_p$ is called an interval of colors. For convenience, for arbitrary integers a,b (not necessarily between 0 and *p*). Let $[a,b]_p=\left[[a]_p,[b]_p\right]_p$. The intervals $(a,b)_p,(a,b]_p,[a,b)_p$ are defined similarly. The length $l([a,b])_p$ of an interval $[a,b]_p$ is the number of integers in the interval and is equal to $[b-a]_p+1$.

If the integer *p* is clear from the context, then we write $[a,b]$ for $[a,b]_p$. When considering (p,q)-colorings of graphs, we say two colors i,j are adjacent if $q\le|i-j|\le p-q$. For two sets A,B of colors, let $A+B=\{[a+b]_p:a\in A,b\in B\}$.

Observe that when considering (p,q)- colorings of graphs, for a set *A* of colors, $A+[q,p-q]_p$ is the set of colors which is adjacent to at least one color in *A*.

Lemma:

Suppose *B* is an interval of colors. For any set *A* of colors

$$|A+B|\ge\min\{|A|+|B|-1,p\}.$$

Proof:

Suppos $A=[a_1,b_1]\cup[a_2,b_2]\cup\cdots\cup[a_t,b_t]$ and $B[c,d]$.The intervals $(b_1,a_2),(b_2,a_3),\cdots,(b_t,a_1)$ are the “gaps” of *A*. It is known (See [8]) that,

$$A+B=\{0,1,\cdots,p-1\}$$

Or

$$A+B=[a_1+c,b_1+d]\cup[a_2+c,b_2+d]\cup\cdots\cup[a_t+c,b_t+d]$$

If there is a gap, say (b_1,a_2) of size at least $[B]$, then $[a_1+c,b_1+d],[a_2+c,b_2+d],\cdots,[a_t+c,b_t+d]$ are pair-wise distinct subsets of Therefore,

$$|A+B|\geq\|[a_1+c,b_1+d]\|+\|[a_2+c,b_2+d]\|+\cdots+\|[a_t+c,b_t+d]\|$$

$$=|A|+|B|-1$$

If each of the gaps of *A* has size less than *B*, then it is easy to see that $A+B=\{0,1,\cdots,p-1\}$ and hence $|A+B|=p$.

Hence the proof.

Theorem:

Suppose *T* is a tree, $p\geq 2q$ are positive integers and $l:V(T)\to\{0,1,2,\cdots,p\}$ is a color-size-list. Then *T* is $l-(p,q)$-colorable if and only if for each subtree T' of *T*. $\sum_{v\in T'} l(V)\geq 2(|V(T')|-1)q+1$.

Proof:

The "only if" part of theorem follows from the following lemma.

Lemma:

Suppose *l* is a color-size-list of a tree $T=(V,E)$. If $\sum_{x\in T} l(x)<2(|V|-1)q+1$, then there is a color-list *L* such that $L(x)$ is an interval of colors with $|L(x)|=l(x)$ for each vertex *x* and *T* is not $L(p,q)$-colorable.

Proof:

We prove lemma by induction on $|V|$. If $V=\{u\}$ then the condition says that $l(v)=0$, and hence $L(v)=\phi$ for the only vertex *v* of *T*. Then of course *T* is not $L-(p,q)$-colorable.

Assume $|V|\geq 2$. Let *v* be a leaf of *T*.

Let *u* be the neighbor of *v*.

If $l(u)+l(v)\leq 2q$, then $L(v)=[0,l(v)-1]_p$ and let $L(u)=[l(v)+p-q,l(v)+p-q+l(u)-1]_p$ and for $x\neq u,v$.

Let $L(x)$ be any interval of colors for which $|L(x)|=l(x)$. Observe that no color in $L(u)$ is adjacent to a color in $L(v)$.

So *T* is not $L-(p,q)$-colorable.

Assume $l(u)+l(v)\geq 2q+1$.

If $l(v)\geq 2q$, then let l' be the color-size-list of $T-v$, defined as $l'(x)=l(x)$ for all *x*.

If $l(v)\leq 2q-1$, then l' be the color-size-list of $T-v$ defined as $l'(x)=l(x)$ if $x\neq u$, and $l'(u)=l(u)+l(v)-2q$

In any case, $\sum_{x\in T-v} l'(v)\leq\sum_{x\in T} l(x)-2q$.

Therefore l' satisfies the condition of lemma.

By induction hypothesis, there is a color-list L' such that $L'(x)$ is an interval of size $l'(x)$ for each vertex *x*, and $T-v$ is not $L'-(p,q)$ colorable. Assume

$L'(u)=[c,d]$. If $l(v)\geq 2q,$ then let *L* be any extension of L'. Any $L-(p,q)$-coloring induces an $L'-(p,q)$ coloring of $T-v$.

Therefore, *T* is not $L-(p,q)$-colorable.

If $l(v)\leq 2q-1$, then let $L(v)=[c-q,c+l(v)-q-1]$, $L(u)=[d-l(u)+1,d]$ and $L(x)=[L'(x)]$ for $x\neq u$, *v*.

Observe that $L(v)+[q,p-q]=[c,c+l(v)-p-2q-1]$.

Since $|[c,d]|=l(u)+l(v)-2q$, we conclude that $(L(v)+[q,p-q])\cup L(v)=[c,d]$.

Therefore if ϕ is an $L-(p,q)$-coloring of *T* such that $\phi(x)\in L(x)$ for all *x*, then $\phi(u)\in[c,d]$, that is, the restriction of ϕ to $T-v$ is an $L'(p,q)$-coloring of $T-v$.

Contrary to the assumption that $T-v$ is not $L'(p,q)$-colorable.

Therefore *T* is not $L-(p,q)$-colorable.

Hence the proof.

The "if" part of theorem follows from the lemma.

Lemma:

Assume *L* is a color-list of *T*. If for each subtree T' of *T*.

$\sum_{v\in T'}|L(v)|\geq 2(|V(T')|)q+1$ then *T* is $L-(p,q)$-colorable.

Proof:

We prove lemma by induction on $|V(T)|$. Assume *L* is a color-list of *T* such that for each subtree T' of T.

$$\sum_{v\in T'}|L(v)|\geq 2(|V'|-1)q+1$$

If $|V(T)|=1$, then the condition implies that $L(v)\neq\phi$ for the only vertex *v* of *T*. Hence *T* is $L-(p,q)$-colorable. Assume $|V(T)|\geq 2$. Let *v* be a leaf of *T*.

Let *u* be the neighbor of *v*. Consider the edge $e=uv$, which is a subtree of *T*.

The condition of lemma implies that $|L(u)|+|L(v)|\geq 2q+1$.

Similarly, as before $L(v)+[q,p-q]_p$ is the set of colors each of which is adjacent to atleast one color of $L(v)$.By lemma (1),

$$|L(v)+[q,p-q]_p|\geq \min\{|L(v)+p-2q,p|\}.$$

If $|L(v)+[q,p-q]_p|=p$, then let L' be the restriction of *L* to $T-v$.

Any $L'-(p,q)$-coloring of ϕ of $T-v$ can be extended to an $L-(p,q)$-coloring of *T*.

Otherwise, $|L(v)+[q,p-q]_p|\geq|L(v)|+p-2q$. Let L' be the color-list of $T-v$ defined as $L'(x)=L(x)$ for $x\neq u$ and $L'(u)=L(u)\cap(L(v)+[q,p-q]_p)$

Then $|L'(u)|\geq|L(u)|+|L(v)|-2q$. Straightforward calculation shows that L' satisfies the condition of lemma.

Therefore $T-v$ has an $L'-(p,q)$-coloring ϕ.

As $\phi(u)\in L'(u)\subseteq L(v)+[q,p-q]_p$, so $\phi(u)$ is adjacent to some color in $L(v)$. Hence ϕ can be extended to an $L-(p,q)$ coloring of *T*.

Hence the proof.

Theorem:

Given a tree *T*, positive integers $p\geq 2q$ and a color-size-list *l* for *T*, it can be determined in linear time whether or not *T* is $l-(p,q)$-colorable.

Proof:

Let *v* be a leaf vertex of *T* and let *u* be the unique neighbor of *v*.

If $l(u)+l(v)\leq 2q$, then *T* is not $l-(p,q)$-colorable by theorem 5.3.2.

Assume $l(u)+l(v)\geq 2q+1$.

Delete *v*, and let $l'(u)=l(u)+l(v)-2q$ and $l'(x)=l(x)$ for $x\neq u,v$.

It follows from theorem (5.3.2) that *T* is $l-(p,q)$-colorable if and only if $T-v$ is $l'(p,q)$-colorable. By repeatedly deleting leaf vertices of *T*, one determines in linear time whether or not *T* IS $l-(p,q)$ colorable.

Hence the proof.

Coloring the Cycles:

We consider list coloring of cycles. Given a cycle $X=(x_0,x_1,\cdots,x_{n-1})$ the vertices are also considered as cyclically ordered. The additions on the indices of the vertices of the cycle are modulo *n*. The intervals $[i,j]_n,(i,j)_n,[i,j)_n,(i,j]_n$ are defined in the same way as the intervals of color.

The following result in the main theorem of this section.

Theorem:

Let $k\geq 1$ be an integer, and $X=(x_0,x_1,\cdots,x_{n-1})$ be a cycle of length $n\geq 2k+1$. Suppose $l:V(X)\to\{0,1,\cdots,2k+1\}$ is a color-size-list for *X*.

Then *X* is $l-(2k+1,k)$-colorable if the following conditions hold.

1. For each interval $[j,j']_n$ of length *m*, $\sum_{t\in[j,j']_n} l(x_t)\geq 2(m-1)k+1$.
2. $\sum_{t=0}^{n-1} l(x_t)\geq 2nk+1$.

Moreover, condition (1) is necessary for *X* to be $l-(2k+1,k)$-colorable, and in case *X* is an odd cycle, condition (2) is sharp.
The necessity of condition 1 follows from lemma because if *X* is $l-(2k+1,k)$-colorable, then each subtree (which is a path) must be $l-(2k+1,k)$-colorable.

If $X=(x_0,x_1,\cdots,x_{n-1})$ is an odd cycle, then condition (2) is sharp in the following sense.

There is a color-size-list *l* which satisfies condition (1) and $\sum_{t=0}^{n-1} l(x_t)=2nk$ However *X* is not $l-(2k+1,k)$-colorable.

For example, if $L(x_i)=[1,2k]$ for each *i*, then $l(x_i)=|L(x_i)|$ satisfies condition (1) and $\sum_{t=0}^{n-1} l(x_t)=2nk$.

However, X is not $L-(2k+1,k)$-colorable, because an $L-(2k+1,k)$-coloring ϕ of X is equivalent to a homomorphism from X to $C_{2k+1}-\{0\}$ and $C_{2k+1}-\{0\}$ is a bipartite graph.

However, condition (2) is not a necessary condition. There are color-size-list l which violates condition (2) and yet X is $l-(2k+1,k)$-colorable.

For example, Suppose $X=(x_0,x_1,x_2,x_3,x_4)$ is a 5-cycle, let $l(x_0)=3,\ l(x_1)=5$ and let $l(x_i)=4$ for $i\geq 2$.

Then X is $l-(5,2)$-colorable, although condition (2) is violated.

Theorem:

If (X,F) is a valid FCA, then there is a good $(2k+1,k)$–coloring for (X,F).

We shall be only considering $(2k+1,k)$-colorings of graphs. For simplicity, we refer a $(2k+1,k)$-coloring simply as a coloring.

Given a FCA, let

$$\Gamma_F=\{(i,j):0\leq i\leq n-1,\ 0\leq j\leq 2k,\ j\in F(i)\}$$

Given a coloring ϕ of X, let

$$\Gamma_\phi=\{(i,j):0\leq i\leq n-1,\ 0\leq j\leq 2k,\ j=\phi(x_i)\}$$

To prove theorem we need to find a coloring ϕ such that $\Gamma_\phi\cap\Gamma_F=\phi$.

It is helpful to have a picture for the understanding of the proof below:

We construct a graph G whose vertex set is partitioned into n coloumns

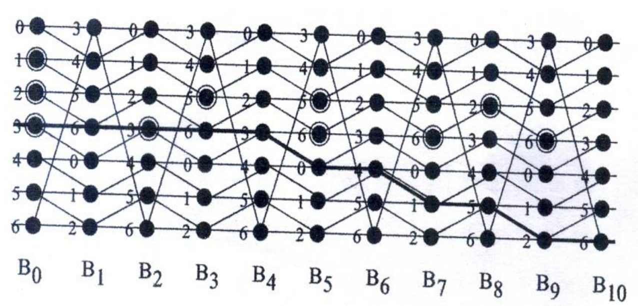
$$B_i=\{(i,j):\ 0\leq j\leq 2k;\ \text{for } i=0,1,\cdots,n-1\}$$

each vertex (i,j) in B_i is connected to two vertices in B_{i+1}, namely $(i+1,j+k)$ and $(i+1,j+k+1)$, where the summation in the first coordinate is modulo n, and the summation in the second coordinate is modulo $2k+1$.
A coloring ϕ corresponds to a cycle of G which intersects each column B_i exactly once.

We call such a cycle of G a “coloring cycle”. The set Γ_F is the set of forbidden vertices in G. We need to find a “coloring cycle” which avoids the forbidden vertices Γ_F. Figure 1 below is an example of the graph G with $k=3$ and $n=11$.

There are edges between vertices in B_{10} and B_0, however, for simplicity, these edges are not shown in the figure.

The thick edges indicates a coloring cycle.



(The two ends should meet, i.e. the vertex 6 in column B_{10} is adjacent to the vertex 3 in column B_0) circled vertices indicate vertices in F, that is,

$$F_0=\{1,2,3\},\ F_2=\{3\},\ F_3=\{5\},\ F_5=\{5,6\},$$
$$F_7=\{6\},\ F_8=\{2\},\ F_9=\{6\},\ F_1=F_4=F_6=F_{10}=\phi$$

Observe that the coloring cycle indicated by the thick edge in figure 1 intersects the “forbidden vertices”. So this coloring is not a good coloring.

We need to define some notations so that we can talk about the "shape" of the set of forbidden vertices.

Suppose (X,F) is a valid FCA, where $X=(x_0,x_1,\cdots,x_{n-1})$ we say a column B_i is infected if B_i contains at least one forbidden vertex, that is $F(i)\neq\phi$.

We say a column B_i is seriously infected if B_i contains at least two forbidden vertices, that is, $|F(i)|\geq 2$.

Let J_F be the set of indices of the infected columns, and let I_F be the set of indices of the seriously infected columns, that is, $J_F=\{0\leq i\leq n-1;\ F(i)\neq\phi\}$. $I_F=\{i:|F(i)|\geq 2\}$ Let $|F|=\sum_{i=0}^{2k}|F(i)|$.

For $i\in J_F$, let q_i^F be the smallest positive integer such that $i+q_i^F\in J_F$. For $i\in I_F$, let P_i^F be the smallest positive integer such that $i+P_i^F\in I_F$.

We shall prove that for a counter example, $I_F\neq\phi$ (lemma 5.4.7) Assume $i\in I_F$ and $[i,i+p_i^F]\cap J_F=\{\beta_0^i,\beta_1^i,\cdots,\beta_{t_i}^i\}$.

Where

$$\beta_0^i=i,\beta_{t_i}^i=i+p_i^F \text{ and for } 1\leq j\leq t_i-1$$
$$\beta_j^i\in(\beta_{j-1}^i,\beta_{j-1}^i)_n$$

Let, $$S_i^F=\sum_{j=0}^{t_j-1} j\times l\left(\left[\beta_j^t,\beta_{j+1}^t\right]_n\right)$$

For the example in figure 1, the corresponding parameters for this FCA are

$$J_F=\{0,2,3,5,7,8,9\},\ I_F=\{0,5\}$$
$$q_0^F=2,q_2^F=q_3^F=1,q_5^F=2,q_7^F=q_8^F=1,q_9^F=2,\ p_0^F=5,\ p_5^F=6,$$
$$\beta_0^0=0,\beta_1^0=2,\beta_2^0=3,\beta_3^0=5,\beta_0^5=5,\beta_1^5=7,\beta_2^5=8,\beta_3^5=9,\beta_4^5=0$$
$$S_0^F=0\times2+1\times1+2\times2=5,\ S_5^F=0\times2+1\times1+2\times1+3\times2=9$$

Definition:

Suppose (X,F) and (X',F') are two FCA's. If the existence of a good coloring for (X',F') implies the existence of a good coloring for (X,F), then we say (X',F') dominates (X,F).

Lemma: If $n=2k+2$ then the conclusion of theorem holds.

Proof:

Without loss of generality, we assume that $0\notin F(0)$ and $1\in F(0)$.

Let ϕ be the coloring defined as $\phi(x_i)=1$ if *i* is even and $\phi(x_i)=k+1$ if *i* is odd. If $\Gamma_F\cap\Gamma_\phi=\{(0,1)\}$, then let $\phi'(x_i)=\phi(x_i)$ for $i\neq 0$ and $\phi'(x_0)=0$.

As $0\notin F(0)$, ϕ' is a good coloring. Assume $|\Gamma_F\cap\Gamma_\phi|\geq 2$, for $i=0,1,\cdots,2k$ let ϕ_i be the coloring of *X* defined as $\phi_i(x)=\phi(x)+i$. Since $|\Gamma_F\cap\Gamma_\phi|\geq 2$ and $|\Gamma_F|\leq 2k+1$, there is an index *i*, such that, $\Gamma_{\phi_i}\cap\Gamma_F=\phi$

Lemma:

Suppose for some index *i*, $F(i)=\{u\}$ and $F(i+1)=\{v\}$. Assume that *u* and *v* are not adjacent. Let $X'=\{x_0',x_1',\cdots,x_{n-3}'\}$ be a cycle of length $n-2$, and let F' be a FCA for X' defined as $F'(j)=F(\tau(j))$, where $\tau:\{0,1,\cdots,n-3\}\to\{0,1,\cdots n-1\}$ is defined as,

$$\tau(j)=\begin{cases} j & \text{if } j\le i-1 \\ j+2 & \text{if } j\ge i \end{cases}$$

Then (X',F') is a valid FCA and dominates (X,F).

Proof:

First we show that (X',F') is valid.

Consider an interval $[j,j']_{n-2}$ of length *m*. If $i\notin[j,j']_{n-2}$ or $i-1\notin[j,j']_{n-2}$ then $[\tau(j),\tau(j')]_n$ also has length *m* and $\sum_{S\in[j,j']_{n-2}}|F'(S)|=\sum_{S\in[\tau(j),\tau(j')]_n}|F(S)|$.

If $i-1$, $i\in[j,j']_{n-2}$, then $[\tau(j),\tau(j')]_n$ has length $m+2$ and

$$\sum_{S\in[j,j']_{n-2}}|F'(S)|=\sum_{S\in[\tau(j),\tau(j')]_n}|F(S)|-2$$
$$\le 2k+m+2-1-2$$
$$=2k+m-1$$

Moreover, $\sum_{S=0}^{n-3}|F'(S)|=\sum_{S=0}^{n-1}|F(S)|-2$
$\le n-1-2\ =n-3$

Therefore (X',F') is valid. Next, we show that (X',F') dominates (X,F).

Let ϕ' be a good coloring for (X',F').

Recall that $F(i)=\{u\}$ and $F(i+1)=\{v\}$

As *u* is not adjacent to *v*, and $\phi'(x'_{i-1})$ is adjacent to $\phi'(x'_i)$.

We conclude that either $\phi'(x'_{i-1})\ne v$ or $\phi'(x'_i)=u$.

If $\phi'(x'_{i-1})\ne v$ then let $t\in\{\phi'(x'_{i-1})+k,\phi'(x'_{i-1})+k+1/\{u\}\}$ and let

$$\phi(x_j)=\begin{cases} \phi'(x'_j) & \text{, if } j\le i-1 \\ t & \text{, if } j=1 \\ \phi'(x'_{j-2}), & \text{if } j\ge i+1 \end{cases}$$

Then ϕ is a good coloring for *F*.

If $\phi'(x_i)\ne u$, then let $t\in\{\phi'(x'_i)+k,\ \phi'(x'_i)+k+1\}/\{v\}$ and let

$$\phi(x_j)=\begin{cases} \phi'(x'_j) & \text{, if } j\le i \\ t & \text{, if } j=i+1 \\ \phi'(x'_{j-2}) & \text{, if } j\ge i+2 \end{cases}$$

Then ϕ is a good coloring for *F*.

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