Modified Finite Element Method for Conductive and Convective Heat Transfer Problems Using Python

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Abstract- *A Modified Finite Element Method (MFEM) for heat transfer problems is developed to get the results very close to that of exact analytical solution. The reciprocal of thermal resistance concept is applied over entire element which results in combined form of analytical and Finite Element Method (FEM). The finite element model for sphere is also developed for conduction through it with convective boundary condition. The MFEM coding has been developed for both conduction and convection in plane wall, cylinder and sphere. The coding is executed and validated in an open source futuristic language Python 3.6. The results obtained from Python is discussed and compared with FEM and analytical methods. The MFEM gives the results exactly equal to that of the exact solutions for plane wall and cylinders, in case of sphere, it is very close to exact solution and better than the results obtained by FEM which gives some approximation and computational errors.*

Keywords- Finite Element Method, Modified FEM, Python 3.6, conduction and convection.

I. INTRODUCTION

It is very difficult to model the physical phenomena in engineering field. The phenomenon for every field in nature can be described by laws of physics or other terms of algebraic, differential and integral equations relating to various quantities of interest. Heat transfer is one of the important practical problems for engineers. The mathematical models are developed using assumptions concerning the appropriate laws governing the process and they are often characterized by complex differential and integral equations posed on geometrically complicated domains. For the past three decades, computers have been used to simplify and solve the practical problems with the help of suitable mathematical models and numerical methods. The computational mechanics is exist for a new and growing body of knowledge connected with the development of mathematical models and use of numerical solutions of physical systems [1].

Finite Element Method (FEM) is one of the important method for numerical simulation of physical problems. The reasons for using FEM are, to find approximate solution using numerical methods for practical problems involve complicated domain, loads, nonlinearities that forbid the development of analytical solutions, cost effective and saves time and material resources compared to physical experiments, and quick process of developing the computer program using powerful electronic computation. The approximate methods for solving differential equations using trial solutions are used by Lord Rayleigh [2], Ritz [3] and Gelerkin [4]. The drawback in their approaches, compared to the modern finite element method, is that the trial functions must apply over the entire domain of the problem. Even though Gelerkin method provides very strong basis for finite element method, Courant [5] introduced the concept of piecewise-continuous function in subdomain. The tem finite element was first used by Clough [6] in 1960. The finite element method was extended to applications in plate bending, shell bending, pressure vessels and elastic structural analysis during 1960s and 1970s.

Wilson and Nickell [7] are applied the finite element approach in heat transfer conduction problems 1966 and Martin [8] applied this concept on fluid flow in 1968. The nonlinear finite element method was focused by Oden and Reddy [9, 10]. The detailed history of finite element method is given by Noor [11]. In recent years extensions and modifications of finite element method have been proposed. These include the Partition of Unity Method (PUM) of Babuska and Melenk [12], the h-p cloud method of Durate and Oden [13] and meshless methods advanced by Belytschko et al. [14].

The present work is focused on developing modified finite element method (MFEM) and finite element model for sphere in conductive and convective heat transfer problems. The modification done on finite element model is presented and this modified approach is applied in the field of conductive and convective heat transfer through plane wall, cylinder, sphere, and composite walls. The comparison of finite element model, modified finite element model and exact analytical solutions has been done. The modified finite element model is developed and coding are created through the computer language Python 3.6. It is an open source software which can be used for simulating the heat transfer problems in easy and quick way. The results obtained from the Python after executions are compared and conclusions are presented.

II. FINITE ELEMENT METHOD

In finite element method the given domain is divided into number of subdomains, called finite element, on each element the approximation functions of weighted-residual can be constructed for the solution of the problem. The finite element method differs from the traditional Ritz, Galerkin, Least-squares, Collocation and other weighted-residual methods in the manner in which the approximation functions are constructed. The basic steps for finite element analysis is given by Reddy [15].

A. Governing equation and boundary conditions

Consider a heat transfer problem, one dimensional steady state without heat generation, of finding the function $T(x)$ that satisfies the differential equation

$$-\frac{d}{dx}\left(kA\frac{dT}{dx}\right)+hPl(T-T_\infty)=0 \qquad \text{for } 0<x<L \tag{1}$$

Take $a = kA$, $c = hPl$, and $f = hPlT_\infty$ then the equation (1) can be written as

$$-\frac{d}{dx}\left(a\frac{dT}{dx}\right)+cT-f=0 \qquad \text{for } 0<x<L \tag{2}$$

the boundary conditions are,

$$T(0)=T_0, \quad \left(a\frac{dT}{dx}\right)_{x=L}=Q_0 \tag{3}$$

where,

$a=a(x), c=c(x), f=f(x)$ and T_0, and Q_0 are the known quantities of the problem.

B. Domain discretization

In finite element method, the domain Ω of the problem shown in Fig.1 (a) is divided into number of subdomains called finite element. A typical element AB is denoted as Ω_e and it is located between the co ordinates x_a and x_b with the element length l_e. The number of elements for a problem depends mainly on the geometry of the domain and accuracy of the solution.

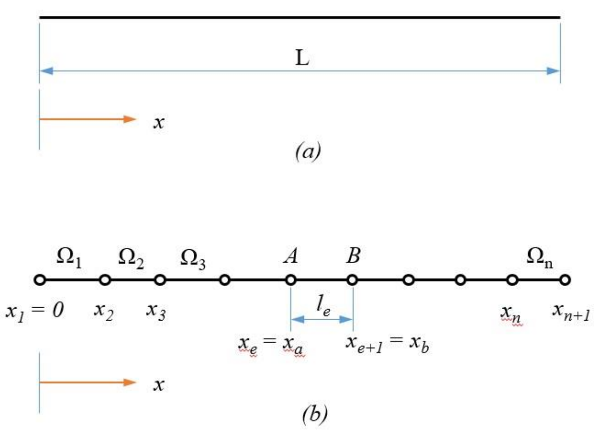


Figure 1. (a) Whole domain, (b) Finite element discretization

C. Element equations

The derivation of finite element equations, i.e., algebraic equations among the unknown parameters of the finite element approximation, involves forming weak form of the differential equation, assuming the form of the approximation solution and substituting the approximate solution into the weak form.

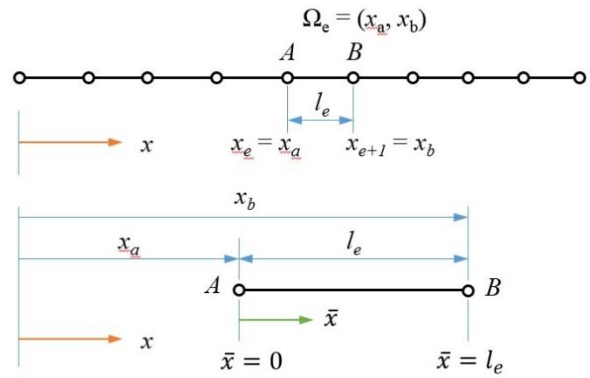


Figure 2. Typical finite element 1D mesh

Step 1. Weak form

The polynomial approximation of the solution for equation (1) over each finite element is given by,

$$T^e = \sum_{j=1}^{n} T_j^e\, \psi_j^e(x) \tag{4}$$

where T_j^e are the values of the solution of $T(x)$ at the nodes of finite element Ω_e and ψ_j^e are the approximation functions over the element. The number of algebraic relations among the T_j^e can be obtained by re arranging the equation (2) in a weighted integral form

$$\int_{x_a}^{x_b} w\left[-\frac{d}{dx}\left(a\frac{dT}{dx}\right)+cT-f\right]dx=0 \tag{5}$$

where $w(x)$ is the weight function and $\Omega_e=(x_a,x_b)$ is the domain of typical element. After multiplying w and integrating we obtain

$$\int_{x_a}^{x_b}\left(a\frac{dw}{dx}\frac{dT}{dx}+cwT-wf\right)dx-\left[wa\frac{dT}{dx}\right]_{x_a}^{x_b}=0 \tag{6}$$

$$Q_1=\left(-a\frac{dT}{dx}\right)_{x_a}, Q_2=\left(a\frac{dT}{dx}\right)_{x_b} \tag{7}$$

After substituting Q_1 and Q_2 in equation (6) the weak form becomes

$$\int_{x_a}^{x_b}\left(a\frac{dw}{dx}\frac{dT}{dx}+cwT-wf\right)dx-w(x_a)Q_1-w(x_b)Q_2=0 \tag{8}$$

Step 2. Approximate solution

For minimum polynomial order of T^e is linear for the weak form approximation,

$$T^e(x)=c_1^e+c_2^e x \tag{9}$$

$$\begin{Bmatrix}T_1^e\\T_2^e\end{Bmatrix}=\begin{bmatrix}1 & x_a\\1 & x_b\end{bmatrix}\begin{Bmatrix}c_1^e\\c_2^e\end{Bmatrix} \tag{10}$$

$$\sum_{i=1}^{n}\psi_i^e(x)=1 \tag{11}$$

The element approximation in terms of coordinate $\bar{x}$ is given by

$$\psi_1^e(\bar{x}) = 1 - \frac{\bar{x}}{l_e}\,,\ \psi_2^e(\bar{x}) = \frac{\bar{x}}{l_e} \tag{12}$$

Step 3. Finite element model

The weak form development of an element with interior nodes is carried out by intervals (x_1^e, x_2^e), $(x_2^e, x_3^e), \ldots\ldots, (x_{n-1}^e, x_n^e)$

$$\sum_{j=1}^{n} K_{ij}^e\, T_j^e - f_i - Q_i = 0 \qquad (i = 1, 2, \ldots, n) \tag{13}$$

$$K_{ij}^e = \int_{x_a}^{x_b} \left(a \frac{d\psi_i^e}{dx} \frac{d\psi_j^e}{dx} + c\psi_i^e \psi_j^e \right) dx \tag{14}$$

$$f_i^e = \int_{x_a}^{x_b} f \psi_i^e\, dx \tag{15}$$

$$\sum_{j=1}^{n} \psi_i^e(x_j^e) Q_j^e = Q_i^e \tag{16}$$

Equations (13) can be written in terms of the coefficients K_{ij}^e, f_i^e, and Q_i^e as

$$\begin{aligned} K_{11}^e\, T_1^e + K_{12}^e\, T_2^e + \cdots + K_{1n}^e\, T_n^e &= f_1^e + Q_1^e \\ K_{21}^e\, T_1^e + K_{22}^e\, T_2^e + \cdots + K_{2n}^e\, T_n^e &= f_2^e + Q_2^e \\ &\vdots \\ K_{n1}^e\, T_1^e + K_{n2}^e\, T_2^e + \cdots + K_{nn}^e\, T_n^e &= f_n^e + Q_n^e \end{aligned} \tag{17}$$

In matrix form the algebraic equations (17) can be written as

$$[K^e]\{T^e\} = \{f^e\} + \{Q^e\} \tag{18}$$

where $[K^e]$ is called coefficient matrix, the column vector $\{f^e\}$ is the source vector, $(T_1^e, T_2^e, \ldots.., T_n^e)$ is primary variables and $Q_1^e, Q_2^e, \ldots.., Q_n^e)$ are the secondary variables.

For linear element,

$$[K^e] = \frac{a_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\ \ \{f^e\} = \frac{f_e l_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{19}$$

Substitute equation (19) in equation (18) we obtain

$$\left(\frac{a_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_e l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} T_1^e \\ T_2^e \end{Bmatrix} = \frac{f_e l_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} + \begin{Bmatrix} Q_1^e \\ Q_2^e \end{Bmatrix} \tag{20}$$

Equation (20) is called as finite element model for linear element.

Connectivity of elements

The equation (20) can be used for e[th] element. Similarly we can express n number of individual elements and finally connect as per the order of their positions. For example of three elements, the matrix form can be written as

$$\begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 + K_{11}^2 & K_{12}^2 & 0 \\ 0 & K_{21}^2 & K_{22}^2 + K_{11}^3 & K_{12}^3 \\ 0 & 0 & K_{12}^3 & K_{22}^3 + K_{11}^4 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ f_2^3 \end{Bmatrix} + \begin{Bmatrix} Q_1^1 \\ Q_2^1 + Q_1^2 \\ Q_2^2 + Q_1^3 \\ Q_2^3 \end{Bmatrix} \tag{21}$$

D. Plane wall

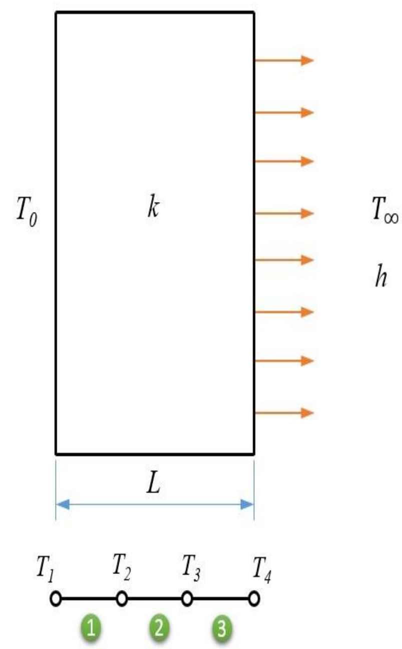


Figure 3. Heat transfer through plane wall

The temperature T_0 at one end of a plane wall is maintained constant and another end is exposed to a fluid at temperature T_∞ with heat transfer coefficient h. The heat is flow from either T_0 to T_∞ or T_∞ to T_0 based on the temperature according to second law of thermodynamics. The plane wall is divided into three number of elements with four number of nodal temperatures T_1, T_2, T_3 and T_4 as shown in Fig.3. The finite element model for this heat transfer problem without heat generation can be expressed as,

$$\left(\frac{a_e}{l_e}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \right) \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ 0 \\ 0 \\ Q_2^3 \end{Bmatrix} \tag{22}$$

E. *Composite wall*

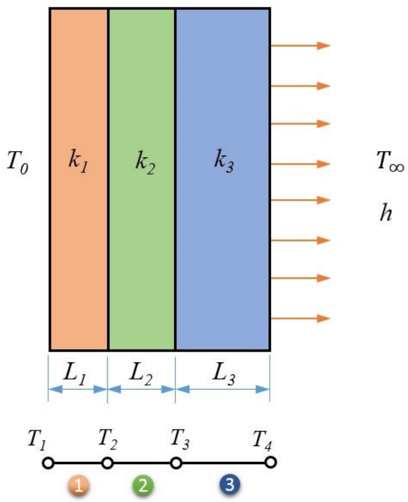


Figure 4. Heat transfer through composite wall

The composite wall consist of three different material with thermal conductivities k_1, k_2 and k_3 with wall thickness L_1, L_2 and L_3 maintained at temperature T_0 at one and exposed another end to a fluid at temperature T_∞ and heat transfer coefficient h is shown in Fig.4. Here the nodal temperatures T_1, T_2, T_3 and T_4 are fixed and will not change. The interior nodal points can be varying with n number of each elements. The finite element model for this heat transfer problem can be expressed as,

$$[K^1] = \frac{a_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_1 l_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[K^2] = \frac{a_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_2 l_2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (23)$$

$$[K^3] = \frac{a_3}{l_3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_3 l_3}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

After assembling the equations (23) can be obtained in the form of (22).

F. *Cylinder*

The cylinder of inner radius r_1 and outer radius r_{n+1}, where n is number of element, thermal conductivity k and length L is maintained at temperature T_0 at inner surface and exposed to a fluid at temperature T_∞ and heat transfer coefficient h is shown in Fig.5.

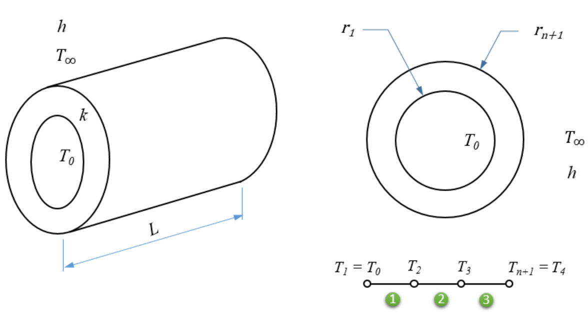


Figure 5. Heat transfer through cylinder

Here T_1, T_2, T_3 and T_4 are nodal temperatures, and r_1, r_2, r_3 an`d r_4 are radius corresponding to $n + 1$ number of nodal points. The finite element model for this heat transfer problem without heat generation can be expressed as,

$$\left(\frac{2\pi k}{l_e}\left(r_a + \frac{h_e}{2}\right)\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\right)\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ 0 \\ 0 \\ Q_2^3 \end{Bmatrix} \qquad (24)$$

G. *Composite cylinder*

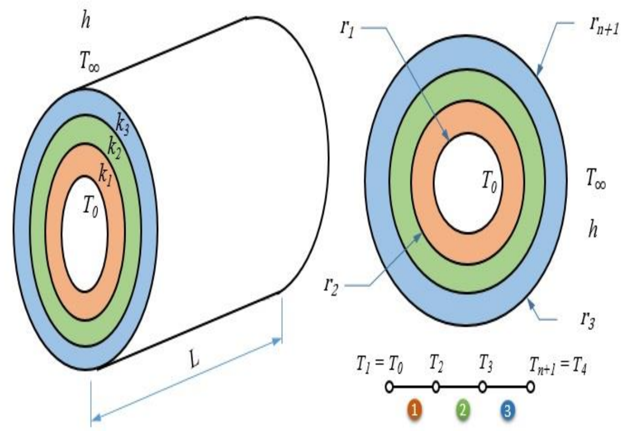


Figure 6. Heat transfer through composite cylinder

The composite cylinder consist of three different material with thermal conductivities k_1, k_2 and k_3 with radius r_1, r_2, r_3 and r_4 maintained at temperature T_0 at inner side and exposed to a fluid at temperature T_∞ and heat transfer coefficient h is shown in Fig.6. Here the nodal temperatures T_1, T_2, T_3 and T_4 are fixed and will not change. The interior nodal points can be varying with n number of each elements. The finite element model for this heat transfer problem can be expressed as,

$$[K^1] = \frac{2\pi k_1}{l_1}\left(r_a + \frac{l_1}{2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[K^2] = \frac{2\pi k_2}{l_2}\left(r_a + \frac{l_2}{2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (25)$$

$$[K^3] = \frac{2\pi k_3}{l_3}\left(r_a + \frac{l_3}{2}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

After assembling the equations (25) can be obtained in the form of (24).

H. Sphere

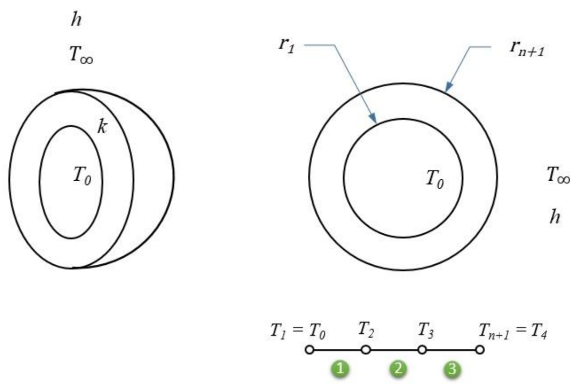


Figure 7. Heat transfer through sphere

The sphere of inner radius r_1 and outer radius r_{n+1}, where n is number of element, thermal conductivity k is maintained at temperature T_0 at inner surface and exposed to a fluid at temperature T_∞ and heat transfer coefficient h is shown in Fig.7. Here T_1, T_2, T_3 and T_4 are nodal temperatures, and r_1, r_2, r_3 and r_4 are radius corresponding to $n+1$ number of nodal points. The finite element model for this heat transfer problem without heat generation can be expressed as,

$$\left(\frac{4\pi k_e}{3l_e}\left((2r_a+l_e)^2-(r_a^2+r_al_e)\right)\begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&1\end{bmatrix}\right)\begin{Bmatrix}T_1\\T_2\\T_3\\T_4\end{Bmatrix}=\begin{Bmatrix}Q_1^1\\0\\0\\Q_2^3\end{Bmatrix} \qquad (24)$$

I. Composite sphere

The composite sphere consist of three different material with thermal conductivities k_1, k_2 and k_3 with radius r_1, r_2, r_3 and r_4 maintained at temperature T_0 at inner side and exposed to a fluid at temperature T_∞ and heat transfer coefficient h is shown in Fig.8.

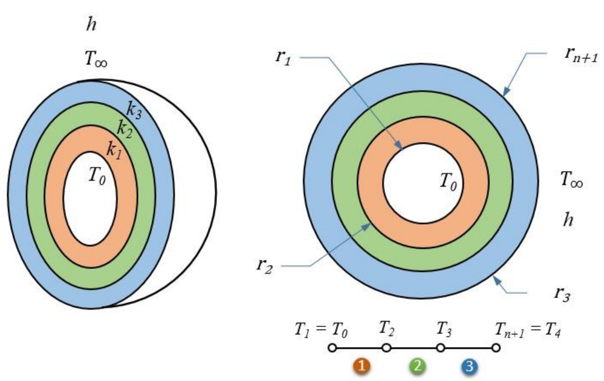


Figure 8. Heat transfer through composite cylinder

Here the nodal temperatures T_1, T_2, T_3 and T_4 are fixed and will not change. The interior nodal points can be varying with n number of each elements. The finite element model for this heat transfer problem can be expressed as,

$$[K^1]=\frac{4\pi k_1}{3l_1}\left((2r_a+l_1)^2-(r_a^2+r_al_1)\right)\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$$

$$[K^2]=\frac{4\pi k_2}{3l_2}\left((2r_a+l_2)^2-(r_a^2+r_al_2)\right)\begin{bmatrix}1&-1\\-1&1\end{bmatrix} \qquad (25)$$

$$[K^3]=\frac{4\pi k_3}{3l_3}\left((2r_a+l_3)^2-(r_a^2+r_al_3)\right)\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$$

After assembling the equations (25) can be obtained in the form of (24).

III. MODIFIED FINITE ELEMENT METHOD

In modified finite element method, the coefficient of stiffness matrix is replaced with the reciprocal of thermal resistance offered against heat flow over each element. The thermal resistance for plane wall, cylinder and sphere are,

A. Plane wall

$$R=\frac{L}{kA} \qquad (26)$$

where,

R = Thermal resistance $\left(\frac{K}{W}\right)$
L= Thickness of the wall (m)
k = Thermal conductivity of the material $\left(\frac{W}{mK}\right)$
A = Area normal to the heat flow (m^2)

B. Cylinder

$$R=\frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi Lk} \qquad (27)$$

where,

R = Thermal resistance $\left(\frac{K}{W}\right)$
r_1 = Inner radius (m)
r_2 = Outer radius (m)
k = Thermal conductivity of the material $\left(\frac{W}{mK}\right)$
L = Length of the cylinder (m)

C. Sphere

$$R=\frac{(r_2-r_1)}{4\pi k r_1 r_2} \qquad (28)$$

where,

R = Thermal resistance $\left(\frac{K}{W}\right)$
r_1 = Inner radius (m)
r_2 = Outer radius (m)
k = Thermal conductivity of the material $\left(\frac{W}{mK}\right)$

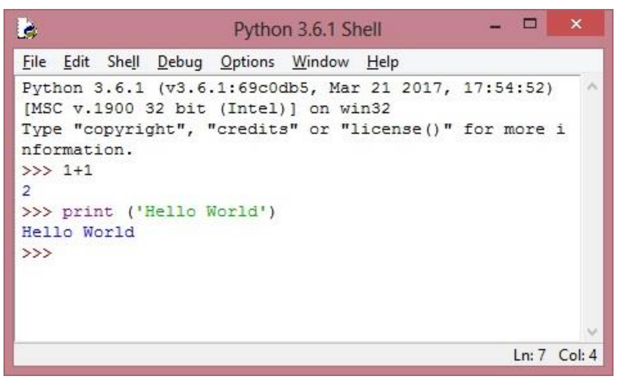
Take $M=1/R$ and the modified finite element model for plane wall without heat generation can be written as,

$$\begin{bmatrix} M & -M & 0 & 0 \\ -M & 2M & -M & 0 \\ 0 & -M & 2M & -M \\ 0 & 0 & -M & M \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} Q_1^1 \\ 0 \\ 0 \\ Q_2^3 \end{Bmatrix} \tag{29}$$

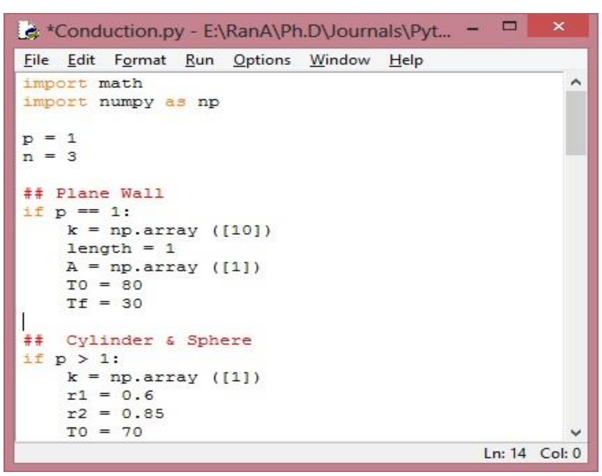
The equation (29) can be called as modified finite element model and it is common for plane wall, cylinder and sphere. The value of M will change corresponding to the type of problems. This form is very simple and easy to determine the unknown temperature at each nodal points. Using this method, the heat transfer through complicated geometries are easily solved with high accuracy as that of analytical solutions.

IV. PYTHON

Python is an open source software is made better when users can easily contribute code and documentation to use it and add features. Python strongly encourages community involvement in improving the software. Python is better and very useful for everyone. The version used for this research work is Python 3.6.1. It consists of different modules and each module can be downloaded freely and installed for creating codes for different works [16]. The coding has been written and executed with the help of number modules like, numpy, scipy, math, etc., The Python UI shell, Python file with extension as .py and Python execution are shown in Fig.9 (a) and (b) respectively.



(a)



(b)

Figure 9. (a) Python shell and (b) Python file

V. RESULTS AND DISCUSSION

The results of three various methods like analytical exact solutions, finite element method solution and modified finite element method solutions are obtained for different geometries and parameters. The obtained results from all the three methods has been compared and suggestions and key features are discussed.

a. Plane wall

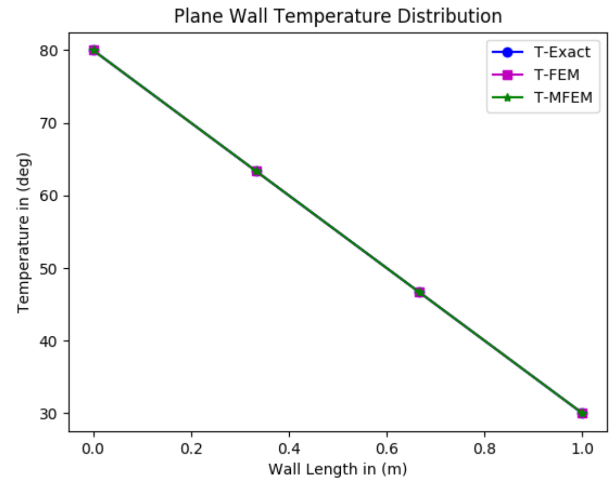


Figure 10. Plane wall Temperature distribution

Consider a plane wall shown in Fig. 3 with thermal conductivity 10 W/m K, temperature at inner and outer surfaces are 80°C and 30°C respectively, unit length and unit area normal to heat flow. Discretize the wall into three elements with equal thickness l_e. Using developed models this heat transfer problem is solved by three different methods, analytical method (Exact), finite element method (FEM) and modified finite element method (MFEM), and the nodal temperatures obtained by each methods are plotted in Fig.10 and the values of nodal temperatures are given with four decimal accuracies in Table 1.

Table 1. Plane wall temperature comparison

Nodal Temperature	Exact Solution	FEM Solution	MFEM Solution
T_1	80	80	80
T_2	63.3333	63.3333	63.3333
T_3	46.6667	46.6667	46.6667
T_4	30	30	30

The heat transfer through plane wall for given conditions gives the nodal temperatures for all the three methods are exactly equal. Here only the variable is nodal temperature T and other parameters like thermal conductivity k, element length l_e, and area normal to heat flow A are remains constant.

b. Cylinder

Consider a cylinder shown in Fig. 5 of inner diameter 0.5 m, outer diameter 2 m, thermal conductivity 10 W/m K, inner and outer surface temperatures are 80°C and 30°C respectively. The heat transfer through the cylinder wall is solved and nodal temperatures are plotted in Fig. 11 and the values are given in Table 2.

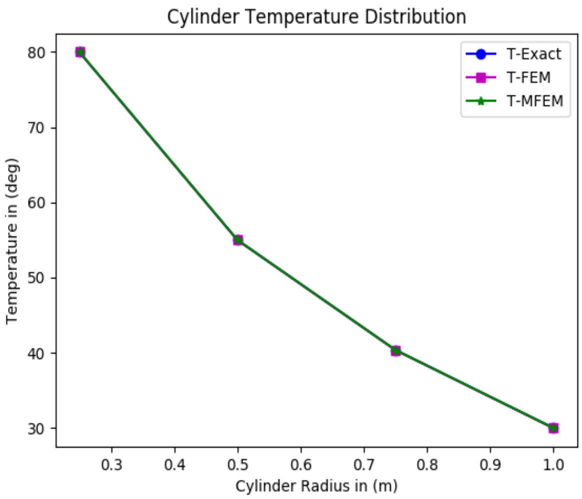


Figure 11. Cylinder Temperature distribution

Table 2. Cylinder temperature comparison

Nodal Temperature	Exact Solution	FEM Solution	MFEM Solution
T_1	80	80	80
T_2	55	55	55
T_3	40.3759	40.3759	40.3759
T_4	30	30	30

In conduction through cylinder wall the nodal temperature are again equal for all the three methods. Though the area normal to the heat flow is varying the nodal temperatures are equal for constant thermal conductivity, element length.

c. Sphere

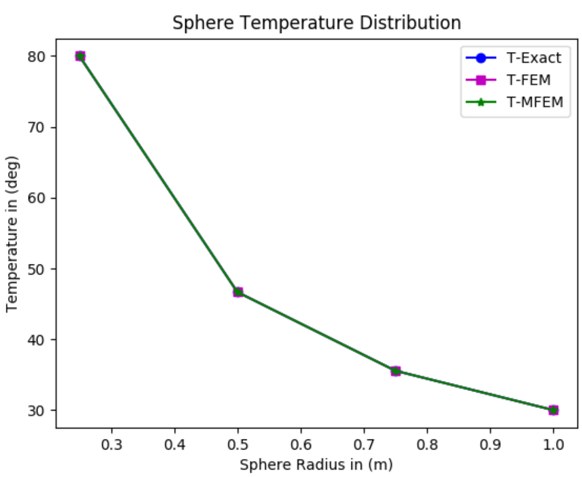


Figure 12. Sphere Temperature distribution

Consider a sphere shown in Fig. 7 of inner diameter 0.5 m, outer diameter 2 m, thermal conductivity 10 W/m K, inner and outer surface temperatures are 80°C and 30°C respectively. The heat transfer through the spherical wall is solved, with help of FEM equation (24) and MFEM equation (29), and nodal temperatures are plotted in Fig. 12 and the values are given in Table 3.

Table 3. Sphere temperature comparison

Nodal Temperature	Exact Solution	FEM Solution	MFEM Solution
T_1	80	80	80
T_2	46.6667	46.6667	46.6667
T_3	35.5556	35.5556	35.5556
T_4	30	30	30

In conduction through spherical wall also the nodal temperatures are exactly equal in all the three methods which shows that in pure conduction without heat generation the nodal temperatures obtained through MFEM produces zero error.

d. Plane wall with convection at one end

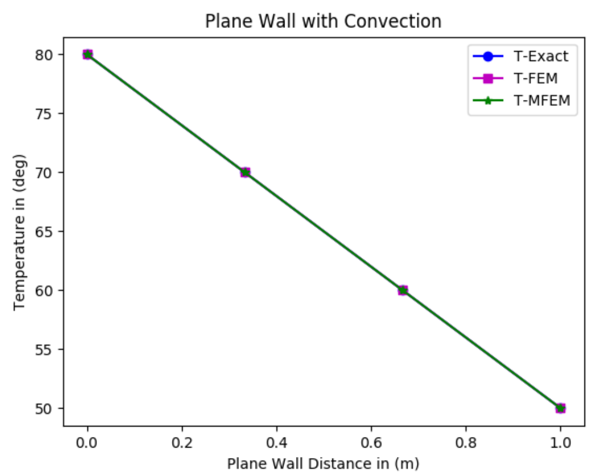


Figure 13. Plane wall with convection at one end

Consider the plane wall shown in Fig.3 contain a convective boundary at one end with heat transfer coefficient h is 15 W/m^2 K and ambient temperature T_∞ is 30°C. This problem is solved and the nodal temperatures are plotted in Fig.13 and the values are given in the Table 4.

Table 4. Plane wall with convection temperature comparison

Nodal Temperature	Exact Solution	FEM Solution	MFEM Solution
T_1	80	80	80
T_2	70	70	70
T_3	60	60	60
T_4	50	50	50

Heat transfer through plane wall with a convective boundary also gives same nodal temperatures for all the three methods and MFEM produces zero error.

e. Cylinder with convection

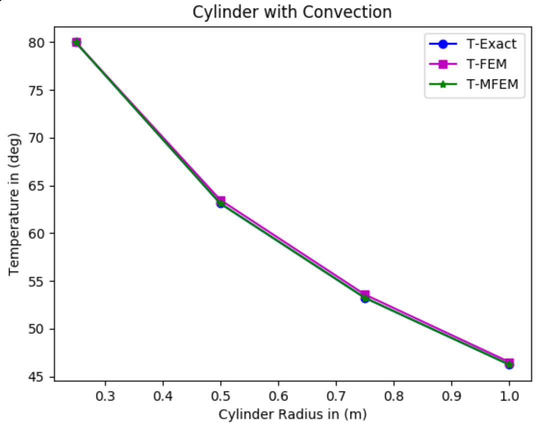


Figure 14. Cylinder with convection at outside

Consider a cylinder shown in Fig. 5 contain a convective boundary at outer side with heat transfer coefficient 15 W/m^2 K and ambient temperature 30°C. The heat transfer through this cylinder is solved and the nodal temperatures are plotted in Fig.14 and the values are given in Table 5.

Table 5. Cylinder with convection temperature

Nodal Temperature	Exact Solution	FEM Solution	MFEM Solution
T_1	80	80	80
T_2	63.1184	63.4906	63.1184
T_3	53.2432	*53.5849	53.2432
T_4	46.2367	46.5094	46.2367

From Table 5 it is clearly noted that the nodal temperatures obtained by FEM produces 0.59% error at node 2, 0.64% of maximum error (*) at node 3 and 0.59% error at node 4 due to approximation and computation but the values obtained through MFEM gives zero error compared to that of exact solution.

f. Sphere with convection

Consider a sphere shown in Fig. 7 contain a convective boundary at outer side with heat transfer coefficient 15 W/m^2 K and ambient temperature 30°C. The heat transfer through this sphere is solved and the nodal temperatures are plotted in Fig.15 and the values are given in Table 6.

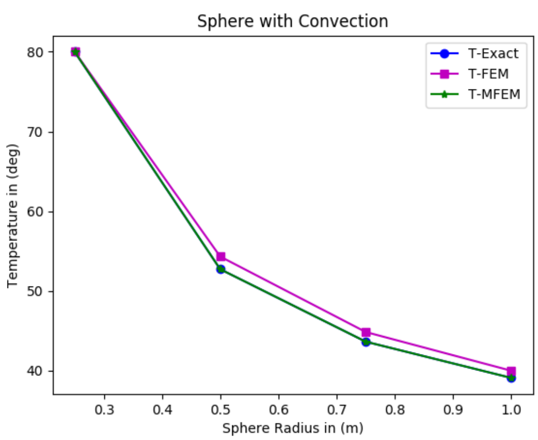


Figure 15. Sphere with convection at outside

From Table 6 it is clearly noted that the nodal temperatures obtained by FEM produces 3.01% of maximum error (*) at node 2, 2.78 % error at node 3 and 2.30% error at node 4 due to approximation and computation but the values obtained through MFEM gives zero error compared to that of exact solution.

Table 6. Sphere with convection temperature comparison

Nodal Temperature	Exact Solution	FEM Solution	MFEM Solution
T_1	80	80	80
T_2	52.7273	*54.3129	52.7273
T_3	43.6364	44.8492	43.6364
T_4	39.0909	39.9894	39.0909

g. Composite wall

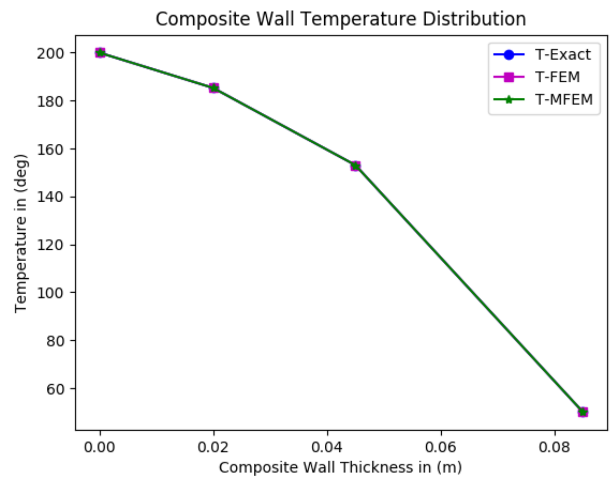


Figure 16. Composite wall

Consider a composite wall consist of three different materials shown in Fig. 4 with thermal conductivities k_1, k_2, k_3 are 70, 40, 20 W/m K respectively, wall thicknesses l_1, l_2, l_3 are 0.02, 0.025, 0.04 m respectively, temperature at inner and outer surfaces are 200°C and 50°C respectively, unit length and unit area normal to heat flow. Using developed models, the nodal temperatures obtained by each methods are plotted in Fig.16 and the values of nodal temperatures are given with four decimal accuracies in Table 7.

Table 7. Composite wall temperature comparison

Nodal Temperature	Exact Solution	FEM Solution	MFEM Solution
T_1	200	200	200
T_2	185.2761	185.2761	185.2761
T_3	153.0675	153.0675	153.0675
T_4	50	50	50

In conduction heat transfer through composite wall the interface temperatures are remains same for all the three methods and the MFEM produces zero error.

h. Composite cylinder

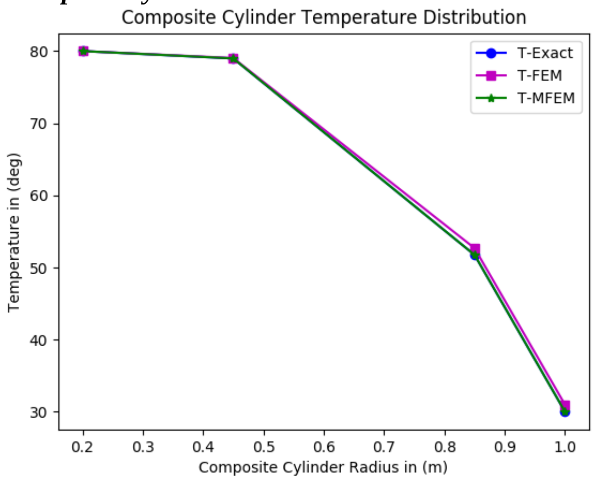


Figure 17. Composite cylinder

Consider a composite cylinder consist of three different materials shown in Fig. 6 of inner diameter 0.4 m, cylinder wall thickness t_1, t_2, t_3 are 0.25, 0.4, 0.15 m respectively, thermal conductivities k_1, k_2, k_3 are 8.5, 0.25, 0.08 W/m K respectively, inner and outer surface temperatures are 80°C and 30°C respectively. The heat transfer through the composite cylinder wall is solved and nodal temperatures are plotted in Fig. 17 and the values are given in Table 8.

Table 8. Composite cylinder temperature

Nodal Temperature	Exact Solution	FEM Solution	MFEM Solution
T_1	80	80	80
T_2	78.9787	79.0312	78.9787
T_3	51.7465	52.6812	51.7465
T_4	30	*30.9825	30

From Table 8 it is clearly noted that the nodal temperatures obtained by FEM produces 0.07% error at node 2, 1.81 % error at node 3 and 3.28% of maximum error (*) at node 4 due to approximation and computation but the values obtained through MFEM gives zero error compared to that of exact solution.

i. Composite sphere

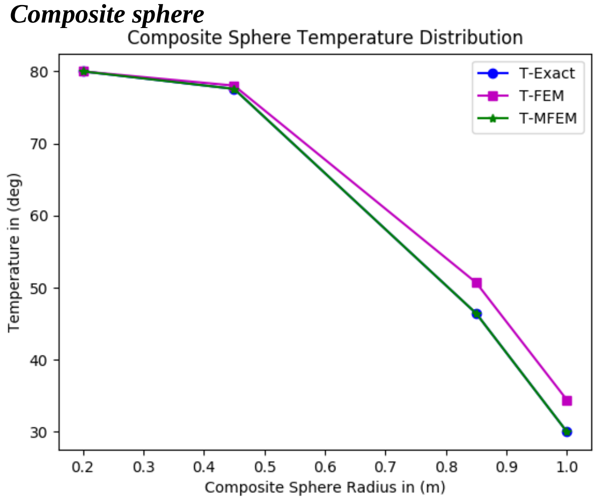


Figure 18. Composite sphere

Consider a composite sphere consist of three different materials shown in Fig. 8 of inner diameter 0.4 m, cylinder wall thickness t_1, t_2, t_3 are 0.25, 0.4, 0.15 m respectively, thermal conductivities k_1, k_2, k_3 are 8.5, 0.25, 0.08 W/m K respectively, inner and outer surface temperatures are 80°C and 30°C respectively. The heat transfer through the composite spherical wall is solved and nodal temperatures are plotted in Fig. 18 and the values are given in Table 9.

Table 9. Composite sphere temperature

Nodal Temperature	Exact Solution	FEM Solution	MFEM Solution
T_1	80	80	80
T_2	77.5669	78.0243	77.5669
T_3	46.4234	50.6918	46.4234
T_4	30	*34.4121	30

From Table 9 it is clearly noted that the nodal temperatures obtained by FEM produces 0.59% error at node 2, 9.19 % error at node 3 and 14.71% of maximum error (*) at node 4 due to approximation and computation but the values obtained through MFEM gives zero error compared to that of exact solution.

j. Composite wall with convection

Consider a composite wall consist of three different materials shown in Fig. 4 with thermal conductivities k_1, k_2, k_3 are 70, 40, 20 W/m K respectively, wall thicknesses l_1, l_2, l_3 are 0.02, 0.025, 0.04 m respectively, temperature at inner surface is 200°C, unit length, unit area normal to heat flow, heat transfer coefficient h is 10 W/m^2 K and ambient temperature T_∞ is 50°C. The nodal temperatures obtained by each methods are plotted in Fig.19 and the values of nodal temperatures are given with four decimal accuracies in Table 10.

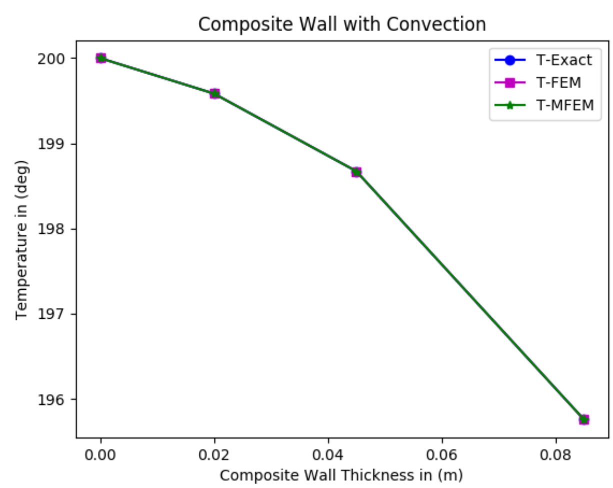


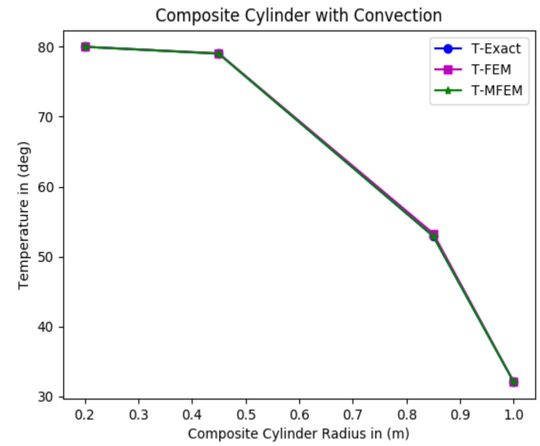
Figure 19. Composite wall with convection

Table 10. Composite wall with convection

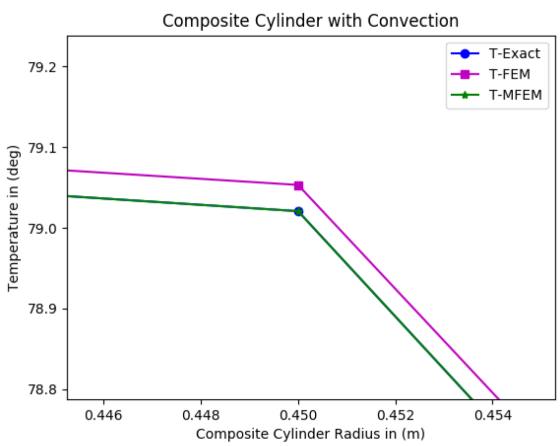
Nodal Temperature	Exact Solution	FEM Solution	MFEM Solution
T_1	200	200	200
T_2	199.5836	199.5836	199.5836
T_3	198.6726	198.6726	198.6726
T_4	195.7574	195.7574	195.7574

In conduction heat transfer through composite wall with convective boundary at one end the interface temperatures are remains same for all the three methods and the MFEM produces zero error.

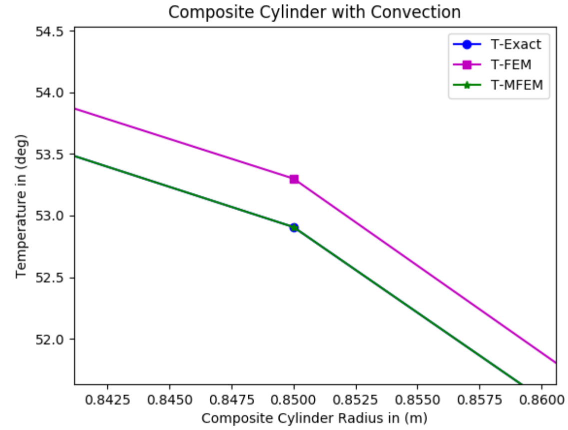
k. Composite cylinder with convection



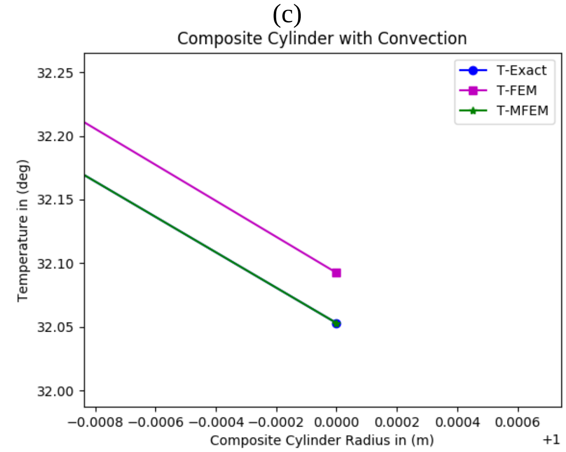
(a)



(b)



(c)



(d)

Figure 20. (a) Composite cylinder with convection, (b) Temperature at node 2, (c) Temperature at node 3 and (d) Temperature at node 4

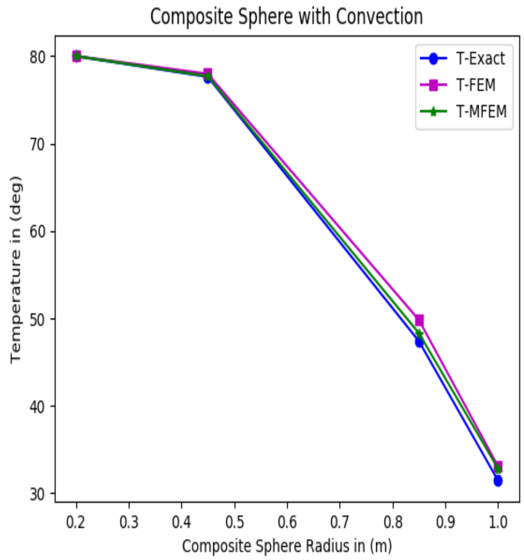
Consider a composite cylinder with convective boundary consist of three different materials shown in Fig. 6 of inner diameter 0.4 m, cylinder wall thickness t_1, t_2, t_3 are 0.25, 0.4, 0.15 m respectively, thermal conductivities k_1, k_2, k_3 are 8.5, 0.25, 0.08 W/m K respectively, inner surface temperatures of 80°C, heat transfer coefficient of 5 W/m^{2}K and ambient temperature of 30°C. The heat transfer through the composite cylinder wall is solved and nodal temperatures are plotted in Fig. 20 and the values are given in Table 11.

Table 11. Composite cylinder with convection temperature comparison

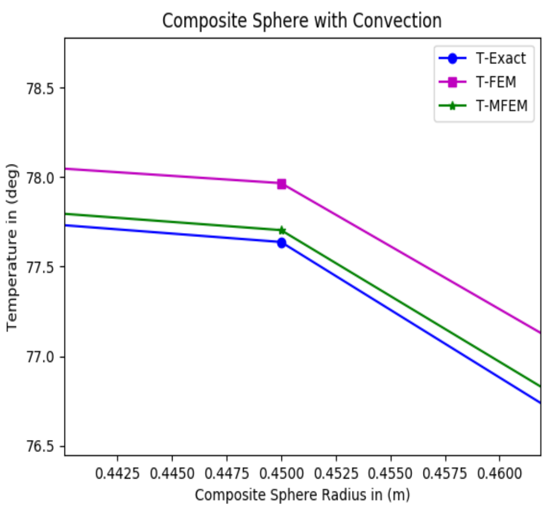
Nodal Temperature	Exact Solution	FEM Solution	MFEM Solution
T_1	80	80	80
T_2	79.0207	79.0532	79.0207
T_3	52.9066	*53.2998	52.9066
T_4	32.053	32.0925	32.053

From Table 11 it is clearly noted that the nodal temperatures obtained by FEM produces 0.04% error at node 2, 0.74 % of maximum error (*) at node 3 and 0.12% error at node 4 due to approximation and computation but the values obtained through MFEM gives zero error compared to that of exact solution.

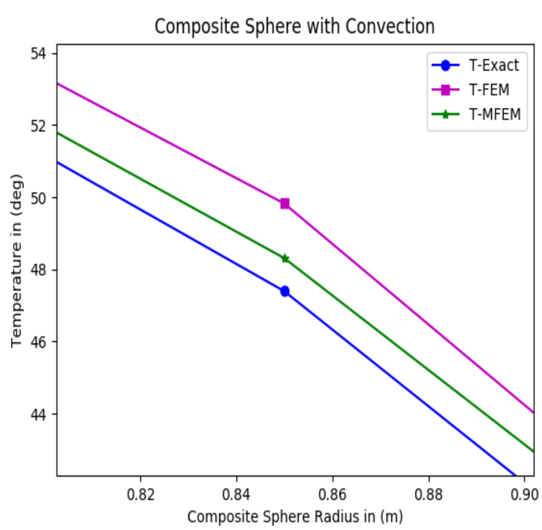
l. Composite sphere with convection



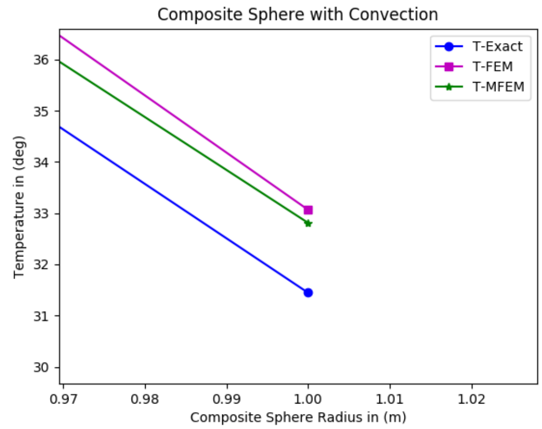
(a)



(b)



(c)



(d)

Figure 21. (a) Composite sphere with convection, (b) Temperature at node 2, (c) Temperature at node 3 and (d) Temperature at node 4

Consider a composite sphere with convective boundary consist of three different materials shown in Fig. 8 of inner diameter 0.4 m, cylinder wall thickness t_1, t_2, t_3 are 0.25, 0.4, 0.15 m respectively, thermal conductivities k_1, k_2, k_3 are 8.5, 0.25, 0.08 W/m K respectively, inner surface temperatures of 80°C, heat transfer coefficient of 5 W/m^{2}K and ambient temperature of 30°C. The heat transfer through the composite spherical wall is solved and nodal temperatures are plotted in Fig. 21 and the values are given in Table 12.

Table 12. Composite cylinder with convection

Nodal Temperature	Exact Solution	FEM Solution	MFEM Solution
T_1	80	80	80
T_2	77.6373	77.9659	77.7037
T_3	47.3944	49.8264	48.3108
T_4	31.446	*33.066	32.8107

From Table 12. it is clearly noted that the nodal temperatures obtained by FEM produces 0.42% error at node 2, 5.13 % error at node 3, 5.15% of maximum error (*) at node 4 due to approximation and computation, and MFEM produces 0.09% error at node 2, 1.93% error at node 3, 4.34% of maximum error (*) at node 4 due to computation compared to that of exact solution.

VI. CONCLUSION

The modified finite element method is successfully developed for conductive and convective heat transfer problems. A finite element model for sphere is also developed and presented for heat transfer application. The models are coded in Python high level language using Python 3.6 open source software consist of different modules freely available in internet. The coding is executed and results obtained for nodal temperatures through three methods, analytical solution, finite element method and modified finite element solution, are discussed for different applications like plane wall, cylinder, sphere and composite walls with and without convective boundary condition. The result shows that the values of nodal temperatures obtained by finite element method produces zero error for plane wall, cylinder, sphere without convective boundary condition and composite wall with boundary condition. In case of cylinder and sphere with convective boundary condition FEM produces some errors due to approximation and computation. The modified finite element method produces zero error for plane wall, cylinder, sphere without convective boundary condition and composite wall, cylinder with convective boundary. It produces the error only for sphere with convective boundary condition and also the error produced by MFEM is lies between the FEM and exact solutions which are very close to exact solutions.

By using this modified finite element method accurate solutions can be obtained by considering the thermal resistance in each and every element and resistance becomes primary variable for heat transfer problem. This method also suitable for complex geometries and uneven discretization of the n number of elements. The coding developed through Python is very easier and quicker than that of FORTRAN, MATLAB and VB.Net. MFEM in Python can produce high accurate results with considerably short duration.

REFERENCES

[1] Research directions in computational mechanics, National academy press, Washington, DC, (1991).

[2] Lord Rayleigh, On the theory of Resonance, Transactions of the Royal Society (London), A161 (1890).

[3] W. Ritz, Uber eine neue Methode zur Losung gewissen Variations-Problems der mathematischen Physik, J. Reine Angew. Math., 135 (1909).

[4] B.G. Galerkin, Series Solutions of Some Problems of Elastic Equilibrium of Rods and Plates [in Russian], Vetn. Inzh. Tekh.,19 (1915).

[5] R. Courant, Variational Methods for the Solution of Problems of Equilibrium and Vibrations, Bulletin of American Mathematical Society, 49 (1943).

[6] R.W. Clough, The Finite Element Method in Plane Stress Analysis, Proceedings of the Second Conference on Electronic Computation, American Society of Civil Engineers, Pittsburgh, 1960.

[7] E.L. Wilson, R.E. Nickell, Application of Finite Element Method to Heat Conduction Analysis, Nuclear Engineering Design, 4 (1966).

[8] H.C. Martin, Finite Element Analysis of Fluid Flows, Proceedings of second conference on Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, Kilborn, Ohio, 1968.

[9] J.T. Oden, Finite Elements of Nonlinear Continua, McGraw-Hill, New York, 1972.

[10] J.T. Oden, J.N. Reddy, An Introduction to the Mathematical Theory of Finite Elements., John Wiley, New York, 1976.

[11] A.K. Noor, Bibliography of Books and Monographs on Finite Element Technology, Applied Mechanics Reviews 44 (6) (1991).

[12] I. Babusk, J.M. Melenk, The Partition of Unity Method, International Journal of Numerical Methods in Engineering, (1996).

[13] C.A. Durate, J.T. Oden, An p-h Adaptive Method Using Clouds, Computer Methods in Applied Mechanics and Engineering, 139 (1996) 237-262.

[14] T. Belytschko, Y. Krongauz, D. Organ, M. Fleming, P. Krysl, Meshless Methods: An Overview and Recent Developments, Computer Methods in Applied Mechanics and Engineering, 139 (1996) 3-47.

[15] J.N. Reddy, An Introduction to the Finite Element Method, McGraw Hill Education (India) Private Limited, New Delhi, 2015.

[16] https://www.python.org