Study on Derivation of The Gravitational Radiation Rate of Linear Approximation

Dr. Noruddin Ansari[1]
[1] Department Of Mathematics
[1] Siwan Engg., & Tech. Institute, Siwan

Abstract- *This paper reveals with the inverse square law for a static source, and the emission of radiation. The difference in their physical meaning and their characteristic form permits one to obtain the total rate of radiation emission (both its momentum rate and its energy rate) from knowledge of the self-force. This can be done without knowledge of the asymptotic form of the radiation field. The electromagnetic case is the main topic, but the method is also applied to the emission of gravitational radiation in the linear approximation to general relativity.*

Keywords- Gravitational mass, General relativity, Linear Approximation, etc.

I. INTRODUCTION

There are only two fundamental interactions in classical physics, the electromagnetic and the gravitational. They have several properties in common, of which the best known are the inverse square law for a static source, and the emission of radiation. The latter phenomenon occurs whenever the source is not in uniform motion.1

That fact establishes that Newton's famous law of motion, $\mathbf{F}_{\text{ext}} = m\mathbf{a}$, cannot be correct because whenever the force does not vanish, $\mathbf{F}_{\text{ext}} \neq 0$, neither does the acceleration, $\mathbf{a} \neq 0$. Therefore, some momentum and some energy of the moving object is lost in the form of radiation. This loss is ignored in $\mathbf{F}_{\text{ext}} = m\mathbf{a}$.. In the present paper, I shall show how this defect is repaired. In particular, I shall demonstrate that, when the loss due to radiation is taken into account, consistency requires that an additional phenomenon be included. That phenomenon is often ignored; its inclusion in the equations of motion gives rise to an additional term, the "Schott term."

When the Schott term is not ignored, it is necessary to distinguish between the self-force and radiation reaction: The former is the sum of the Schott term and the radiation reaction term in the equations of motion. Unfortunately, the physics literature is full of confusion on this point often using the terms "radiation reaction" and "radiation damping" incorrectly. The origin of this confusion will be seen in Sec. II.

I shall discuss primarily electrodynamics. But in the last section, I will try to use the method for finding the rate of radiation emission from the self - force also in the gravitational case. This will be done in the so-called "linear approximation" to general relativity in which the gravitational equations of motion have a striking similarity to electrodynamics.

In order to avoid cluttering up the equations with unnecessary numerical factors that take away attention from the physical meaning, I shall use Gaussian units as in Jackson's text2 (the 2nd edition or the second half of the 3rd edition). I shall also use units in which the speed of light c = 1; an easy dimensional analysis can supply the necessary factors of c whenever they are desired.

II. THE LAD EQUATIONS OF MOTION FOR A CHARGE

In covariant notation, Newton's equations of motion read (I use tr $\eta_{\mu\nu} = +2$),

$$m\dot{v}^{\mu} = F^{\mu}_{\text{ext}}, \qquad (2.1)$$

where the external force can be (but does not have to be) the Lorentz force. Since this equation ignores the emission of radiation, one might be tempted to take it into account by simply subtracting on the right-hand side (since it's a loss) the rate of energy–momentum that is leaving the source. In special relativity, the four-vector rate at which the energy-momentum of radiation is emitted is the generalization of the Larmor formula,

$$dP^{\mu}/d\tau = v^{\mu}\mathfrak{R}, \quad \mathfrak{R} = \tfrac{2}{3}e^2\dot{v}^{\alpha}\dot{v}_{\alpha}. \qquad (2.2)$$

But when this is done, one obtains an inconsistent equation. This can be seen as follows: Both sides of (2.1) are orthogonal to the velocity vector v^{μ}, $\dot{v}^{\mu}v_{\mu} = 0$, and

$F^{\mu}_{\text{ext}} v_{\mu} = 0$, while (2.2) is parallel to it. Therefore, when the augmented equation is multiplied by v^{μ}, one obtains $\mathfrak{R} = 0$. A different approach to correcting (2.1) for radiation emission must therefore be taken.

Since both terms of (2.1) are orthogonal to v^{μ}, the added term must be of the form

$$P^{\mu\nu} X_{\nu} = (\eta^{\mu\nu} + v^{\mu} v^{\nu}) X_{\nu}. \qquad (2.3)$$

The factor $P^{\mu\nu}$ is a projector into the hyper-plane orthogonal to the velocity v^{μ}. Assuming that the new equation will be linear and not higher than of second order in the time derivative of the velocity, X^{μ} must have the form

$$X^{\mu} = a v^{\mu} + b \dot{v}^{\mu} + c \ddot{v}^{\mu}. \qquad (2.4)$$

When $P^{\mu\nu}$ acts on that, the first term vanishes, and one finds $b\dot{v}^{\mu} + cP^{\mu\nu}\ddot{v}_{\nu}$. The first term is just like the inertial term on the left-hand side of the equation. Putting $b = -\delta m$ and combining the two inertial terms yields

$$m_0 \dot{v}^{\mu} = F^{\mu}_{\text{ext}} + cP^{\mu\nu}\ddot{v}_{\nu}. \qquad (2.5)$$

Here, the physical rest mass $m_0 = m + \delta m$. Substitution for $P^{\mu\nu}$ and differentiation by parts yields

$$m_0 \dot{v}^{\mu} = F^{\mu}_{\text{ext}} + F^{\mu}_{\text{self}}, \quad F^{\mu}_{\text{self}} = \tfrac{2}{3}e^2(\ddot{v}^{\mu} - v^{\mu}\dot{v}^{\alpha}\dot{v}_{\alpha}), \quad (2.6)$$

provided we identify the parameter c with $2e^2/3$, a necessity for recovering (2.2). This makes (2.6) identical to the Lorentz–Abraham–Dirac equation. The nonrelativistic limit of that equation was first suggested by Lorentz, then derived for the relativistic case for the first time by Abraham3 in 1904 (the year before Einstein's first paper on special relativity was published and therefore without the benefit of its insight) starting from Maxwell's equations. Finally, it was derived in covariant form for a point charge by Dirac4 in 1938. I believe, therefore, that the usual name "Lorentz–Dirac equation" is unfair to Abraham; I shall call it the Lorentz–Abraham–Dirac equation or LAD equation for short. The derivation given above is of course just a heuristic short-cut and avoids a very long and difficult calculation.

The new term in Newton's equation of motion is the four vector F^{μ}_{self}, which is sometimes called the von Laue four vector because he was the first one to put that term of Abraham's equation into manifestly covariant form.5 It is the self-force because it is due to the charge's own field acting on itself. It consists of two terms: The first term is the Schott term. Schott gave the first unobjectionable derivation of the LAD equation in his book.6 The second term is just the negative of (2.2), the rate at which momentum and energy are lost in the form of radiation. It is therefore properly called radiation reaction , F^{μ}_{self}, Thus we have

$$F^{\mu}_{\text{self}} = F^{\mu}_{\text{Sch}} + F^{\mu}_{\text{rad}}, \qquad (2.7a)$$

$$F^{\mu}_{\text{Sch}} = \tfrac{2}{3}e^2\ddot{v}^{\mu}, \quad F^{\mu}_{\text{rad}} = -\tfrac{2}{3}e^2 v^{\mu}\dot{v}^{a}\dot{v}_{a} = -v^{\mu}\mathfrak{R}. \quad (2.7b)$$

These two force four vectors are easily characterized: F^{μ}_{Sch} contains neither the velocity nor the acceleration, and it is a total time derivative. F^{μ}_{rad}, the radiation reaction, is along the negative direction of the four velocity (for obvious reasons of symmetry) and the invariant total radiation rate, $\mathfrak{R}$, is a positive Lorentz invariant scalar (or vanishes). The self-force, F^{μ}_{self}, can therefore always be separated uniquely into these two components.

Unfortunately, the literature is replete with confusion on these terms. Thus Pauli7 calls F^{μ}_{self} "radiation reaction," while Landau and Lifshitz8 call it "radiation damping." This nomenclature was mindlessly repeated in other texts.1,9

The origin of this confusion can be traced to the nonrelativistic limit. In that limit,

$$\mathbf{F}_{\text{sch}} = \tfrac{2}{3}e^2\ddot{\mathbf{v}}, \quad F^{0}_{\text{Sch}} = \tfrac{2}{3}e^2(\mathbf{v}\cdot\ddot{\mathbf{v}} + \dot{\mathbf{v}}^2)$$

while

$$\mathbf{F}_{\text{rad}} = 0, \quad F^{0}_{\text{rad}} = -\tfrac{2}{3}e^2\dot{\mathbf{v}}^2,$$

so that the equations of motion become

$$m_0\dot{\mathbf{v}} = \mathbf{F}_{\text{ext}} + \mathbf{F}_{\text{self}} \qquad (2.8)$$

where $\mathbf{F}_{\text{self}} = \frac{2}{3}e^2\ddot{\mathbf{v}}$ and

$$\frac{d}{dt}\left(\frac{m_0 v^2}{2}\right) = \mathbf{F}_{\text{ext}}\cdot\mathbf{v} + \mathbf{F}_{\text{self}}\cdot\mathbf{v}. \tag{2.9}$$

The last term is the rate of work done by $\mathbf{F}_{\text{self}}$ and consists of two parts,

$$\frac{2}{3}e^2\ddot{\mathbf{v}}\cdot\mathbf{v} = \frac{2}{3}\frac{e^2}{m_0}\frac{d^2}{dt^2}\left(\frac{m_0 v^2}{2}\right) - \frac{2}{3}e^2\dot{\mathbf{v}}^2. \tag{2.10}$$

Thus the rate of work done by the self-force provides not only for the rate of emission of radiation energy but also contributes a term proportional to the second time derivative of the kinetic energy, a Schott-type term. The latter vanishes when the motion is assumed periodic and one averages over time. This is the case usually considered. In that case, one can call $\mathbf{F}_{\text{self}}$ of (2.8) a "damping force" that produces a loss, namely radiation, even though that energy is supplied by the work due to $\mathbf{F}_{\text{ext}}$ and/or by a loss of kinetic energy. However, in case this assumption does not hold, the Schott-type term in (2.10) is not negligible so that part of the radiation loss comes also from that first term. An extreme case is the famous (or infamous?) radiation from a charge in hyperbolic motion (moving under a constant acceleration).[9] In that case, a relativistic treatment shows that F^{μ}_{self} *vanishes* so that (2.7b) implies that *all* the radiation energy comes from the Schott term.

The Schott force, F^{μ}_{Sch}, is a *total time derivative*. As such, it describes a *reversible* process being able to take on either sign, while F^{μ}_{rad} describes an *irreversible* process involving a *loss* of both momentum and energy. Physically, F^{μ}_{rad} involves only the asymptotic fields, the fields that decrease like $1/r$, while F^{μ}_{Sch} involves all the rest, both $1/r^2$ fields and cross terms (the energy–momentum tensor is bilinear in the fields).

This separation of the self-force into Schott and radiation terms, therefore permits one to deduce F^{μ}_{rad} uniquely: it must be of the form (2.7b) with $\mathfrak{R}>0$. This means that one can find the emitted rate of energy and momentum of radiation directly from F^{μ}_{self} without having to compute first the radiation fields (the "$1/r$ fields"), then the Poynting vector, and then having to integrate the angular distribution, to obtain, finally, the *total* momentum and energy of the emitted radiation. If only the *total* momentum and energy emitted per unit time is needed, and if the self-force is known, a very simple method is thus available: one computes the self-force four vector, and then uniquely obtains the four vector of radiation reaction from that of the self-force. The point is that the self-force requires only the knowledge of the fields at the source and not the asymptotic fields. This is the method that will be used in Secs. III and IV. (One should add that this method becomes more complicated when several charges are involved but this is not the present concern.)

But before leaving the LAD equation, a remark must be made about the unphysical (sometimes called "pathological") solutions of this equation. The reason that this equation has such solutions comes from the approximation made in its derivation: the limit to a point charge is taken. This is physically unjustified because it exceeds the domain of applicability of classical physics.[10] But the LAD equation can easily be "repaired." This was apparently first done by Landau and Lifshitz (Ref. 8, paragraph 76). One observes that F^{μ}_{self} is small compared to the other terms in the equation. One can therefore take the equation neglecting F^{μ}_{self}, that means (2.1), as a first approximation, find v^{μ} and its time derivatives from it, and then substitute this into F^{μ}_{self}. The result is an equation free of unphysical solutions. The deeper mathematical meaning of this approximation can be learned from Kunze and Spohn.[11]

III. ELECTROMAGNETIC RADIATION FROM A SPHERICAL CHARGE

The generalization of the LAD equation to an equation for an extended source has recently been carried out.[12] It is a generalization of the method used before for a sphere with a uniform surface charge.[10] But now the charge distribution is left unspecified except that it is assumed to be spherically symmetric. The calculation is based on one that can be found in Jackson[1] [Eq. (17.28) in the 2nd ed., (16.28) in the 3rd ed.] for the self-force of a sphere in its instantaneous rest frame. A boost to an arbitrary frame yields the desired result.

For a spherically symmetric charge distribution $\rho(x)$ (normalized to $\int \rho(x)d^3x = 1$) and confined to a sphere of radius a, F^{μ}_{self} is in good approximation

$$F^{\mu}_{\text{self}}(e,a) = -\frac{2}{3}e^2[\eta^{\mu\nu} + \nu^{\mu}(\tau)\nu^{\nu}(\tau)]\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}I_n\left(\frac{\partial}{\partial\tau}\right)^n\dot{\nu}_{\nu}(\tau), \quad (3.1)$$

where

$$I_n = \int\int d^3\mathbf{x}d^3\mathbf{x}'\rho(\mathbf{x})|\mathbf{x}-\mathbf{x}'|^{n-1}\rho(\mathbf{x}'). \qquad (3.2)$$

Here, ν^{μ} is the velocity of the center of mass of the sphere. This self-force is the result of the retarded self-interaction within the sphere computed in its instantaneous rest frame. In the point limit, (3.1) reduces to F^{μ}_{self} of (2.7).

When (3.1) is expanded, the n = 0 terms contribute only an inertial term of the form $-\delta m\ \dot{\nu}^{\mu}$ with $\delta m = 2e^2I_0/3$ being the electromagnetic contribution to the mass. When combined with the left-hand side of the equation of motion, this term yields the inertial term $m_0\dot{\nu}^{\mu}$, $m_0 = m + \delta m$ being the observed rest mass. The n = 1 term gives exactly the self-force F^{μ}_{self} of the LAD equation, (2.6), and is independent of the charge distribution since $I_1 = 1$. Thus, only the sum from 2 to ∞ contributes charge distribution dependent terms, yielding

$$m_0\dot{\nu}^{\mu} = F^{\mu}_{\text{ext}} + F^{\mu}_{\text{self}}, \quad F^{\mu}_{\text{self}} = F^{\mu}_{\text{Sch}} + F^{\mu}_{\text{rad}}, \quad (3.3)$$

where

$$F^{\mu}_{\text{Sch}} = F^{\mu}_{\text{Sch}}(\text{LAD}) - \frac{2}{3}e^2\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}I_n\left(\frac{\partial}{\partial\tau}\right)^n\dot{\nu}^{\mu}(\tau) \quad (3.4a)$$

and

$$F^{\mu}_{\text{rad}} = F^{\mu}_{\text{rad}}(\text{LAD}) - \frac{2}{3}e^2\nu^{\mu}(\tau)\nu^{\nu}(\tau)\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}I_n\left(\frac{\partial}{\partial\tau}\right)^n\dot{\nu}_{\nu}(\tau). \quad (3.4b)$$

In the point limit, $a \rightarrow 0$, all the $I_n = 0$ for $n \geqq 2$, and one obtains the LAD result for point charges.

The total invariant radiation rate for an arbitrary (but spherical) charge distribution, is therefore

$$\Re = \frac{2}{3}e^2\dot{\nu}^{\alpha}\dot{\nu}_{\alpha} + \frac{2}{3}e^2\nu^{\alpha}(\tau)\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}I_n\left(\frac{\partial}{\partial\tau}\right)^n\dot{\nu}_{\alpha}(\tau). \quad (3.5)$$

If, in particular, the charge is distributed uniformly over the surface of the spherical source, the I_n can be integrated and the sum can be carried out.[10] One obtains (with $\tau_a = 2a$)

$$\Re = -\frac{1}{3}\frac{e^2}{a^2}\left[\nu^{\alpha}(\tau)\nu_a(\tau-\tau_a) + 1\right], \qquad (3.6)$$

which agrees with (2.7b) in first approximation.

IV. GRAVITATIONAL RADIATION FROM A SPHERICAL MASS

A similar analysis can be made for the case of an extended neutral particle in a gravitational field.[12] In the linear approximation of general relativity,[13] the gravitational effects are treated as a correction to the Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Defining $\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ with $h = \eta_{\mu\nu}h^{\mu\nu}$, one can choose $\gamma_{\mu\nu}$ so that it satisfies

$$\Box\gamma_{\mu\nu} = -16\pi G T_{\mu\nu}. \qquad (4.1)$$

Here, $T_{\mu\nu}$ is the matter tensor. This equation shows great similarity to the equation for the four potential in electrodynamics when the Lorentz gauge is used. But this similarity continues. The geodesic equation is the equation of motion of a freely falling particle in a gravitational field,

$$m\dot{\nu}^{\mu} = -m\Gamma^{\mu}_{\alpha\beta}\nu^{\alpha}\nu^{\beta}. \qquad (4.2)$$

The factors m are gratuitous; they express the equivalence principle: the inertial mass (left-hand side) and the gravitational mass (right-hand side) are equal. $\Gamma^{\mu}_{\alpha\beta}$ is the Christoffel symbol, which involves first derivatives of the metric tensor $g_{\mu\nu}$. When the above substitution is made for it, (4.2) becomes (in the instantaneous particle rest frame)

$$\dot{\mathbf{v}} = -\frac{1}{4}\nabla\gamma - \frac{\partial}{\partial t}\boldsymbol{\gamma}, \quad \gamma = \gamma^0_0, \quad \boldsymbol{\gamma} = (\gamma^{\mu}_0). \quad (4.3)$$

Here is the second striking similarity to electrodynamics. It suggests the notation $A^\mu = -\gamma^{\mu}_{0}/4$ and $E = -\nabla\phi - 4\dot{A}$. The factor 4 and an overall sign are the only differences compared with the electrodynamic case: $m\dot{v} = -mE$; but the mass now replaces the charge.

This similarity of the linear approximation of general relativity to electrodynamics has been exploited in the past. Thus, the Lens–Thirring effect (the dragging of inertial frames by a rotating mass, the analog of the Coriolis-type acceleration due to the self-force exerted by the magnetic field produced by a rotating charge) can be computed in that way (see p. 192 of Ref. 13). This suggests that the above calculation of the electromagnetic self-force can be repeated for the gravitational case. The only change would be a factor 4 wherever the vector potential appears. One finds exactly the same result for the gravitational self-force as in the electromagnetic case except for an overall numerical factor,

$$F^{\mu}_{\text{self}}(G) = \frac{11}{3} Gm^2[\eta^{\mu\nu} + v^{\mu}(\tau)v^{\nu}(\tau)]\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} I_n\left(\frac{\partial}{\partial\tau}\right)^n \dot{v}_{\nu}(\tau). \quad (4.4)$$

The new overall factor has the opposite sign from that for the electromagnetic case, (3.1). The reason is very simple: equal charges repel, while masses attract one another.

As in the electromagnetic case, the first two terms of the sum deserve special consideration. The n = 0 term is again an inertial term, $+\delta m\dot{v}^{\mu}$ with $\delta m = 11Gm^2I_0/3$. Note that this gravitational mass gives a negative contribution, $m_0 = m - \delta m$, in contrast to the electromagnetic case. Even more interesting is the fact that such a "mass renormalization" violates the principle of equivalence! The inertial mass is renormalized while the gravitational mass is not. I have elaborated on this elsewhere.[14]

The n = 1 term in the sum (4.4) gives a contribution analogous to the self-force in the LAD equation,

$$F^{\mu}_{\text{self}}(\text{HG}) = -\frac{11}{3} Gm^2(\ddot{v}^{\mu} - v^{\mu}\dot{v}^{\alpha}\dot{v}_{\alpha}), \quad (4.5)$$

the label HG indicating that this result was first derived by Havas and Goldberg.[15]

Thus, one can write the gravitational equation of motion exactly as in the electromagnetic case, Eq. (3.3),

$$F^{\mu}_{\text{Sch}} = F^{\mu}_{\text{Sch}}(\text{HG}) + \frac{11}{3} Gm^2 \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} I_n \left(\frac{\partial}{\partial t}\right)^n \dot{v}^{\mu}(\tau), \quad (4.6a)$$

$$F^{\mu}_{\text{rad}} = F^{\mu}_{\text{rad}}(\text{HG}) + \frac{11}{3} Gm^2 v^{\mu}(\tau)v^{\nu}(\tau)\sum_{n=2}^{\infty} \frac{(-1)^n}{n!} I_n\left(\frac{\partial}{\partial\tau}\right)^n \dot{v}_{\nu}(\tau). \quad (4.6b)$$

Here,

$$F^{\mu}_{\text{Sch}}(\text{HG}) = \frac{11}{3} Gm^2\ddot{v}^{\mu}, \quad F^{\mu}_{\text{rad}}(\text{HG}) = \frac{11}{3} Gm^2 v^{\mu}\dot{v}^{\alpha}\dot{v}_{\alpha}, \quad (4.7)$$

in analogy to (2.7b). But now the radiation rate has the wrong sign! This was of course also noted by Havas and Goldberg. The explanation lies in the linear approximation. In that approximation, the equations "don't know" that gravitational radiation cannot be emitted as dipole radiation but only as quadrupole radiation or as higher multipoles. In fact, this can be seen from the above derivation: from the equation of motion (4.1) only the γ^{μ}_{0} and not the γ_{kl} terms were used. Thus the term F^{μ}_{rad} (HG) should simply be ignored.

Can the correct gravitational radiation rate be obtained from the higher terms in the expansion? To answer this question, I make the following argument. The next term in the series cannot be expected to give gravitational radiation since that next term would be the cross term between the dipole and the quadrupole radiation. This leads to consideration of the n = 3 term of the sum,

$$-\frac{11}{18} Gm^2 I_3 v^{\alpha}\left(\frac{d^4}{d\tau^4} v_{\alpha}\right)v^{\mu}.$$

This term contains total time derivatives similar to the Schott term. These must be separated from the radiation reaction term; since the above inner product of four vectors can be written as $\ddot{v}^{\alpha}\ddot{v}_{\alpha}+$ (total derivatives), the rate of energy–momentum four-vector emission of gravitational radiation is

$$F^{\mu}_{\text{rad}}(G, \text{QUAD}) = -\frac{11}{18} Gm^2 I_3 \ddot{v}^{\alpha}\ddot{v}_{\alpha}v^{\mu}. \quad (4.8)$$

In analogy to the electromagnetic case, the invariant gravitational radiation rate therefore follows as

$$\mathfrak{R}(G) = \frac{11}{18} Gm^2 I_3 \ddot{v}^{\alpha}\ddot{v}_{\alpha}. \quad (4.9)$$

Since $v^{\alpha} = dx^{\alpha}/d\tau$, we see that it is proportional to the sixth power of frequency, as it should be but (4.9) is quite different from the standard result that involves the square of the quadrupole moment (see, e.g., Ref. 8, Sec. 110). Furthermore, $\mathfrak{R}$ should be positive definite, and that holds only if in the instantaneous rest frame $|\ddot{\mathbf{v}}|>|\dot{\mathbf{v}}|^2$. Such an inequality cannot be assured for general motion; it also violates the assumption made in the derivation of (4.8) according to which higher time derivatives are small compared to lower ones. It follows that (4.9) cannot be taken seriously and the derivation of the gravitational radiation rate fails in this linear approximation.

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