Study of Multiple Boolean Algebras

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Abstract- *In this paper the concept of 'Multiple Boolean Algebras' is studied and power set of Multiple Boolean Algebras is introduced.*

Keywords- Multiple Boolean Algebra, Fuzzy set, Boolean Algebra

I. INTRODUCTION

As an outgrowth of introduction of Fuzzy Sets by L. A. Zadeh in 1965, Silvano Di Zenzo[1] introduced the concept of Multiple Boolean algebra. A Multiple Boolean Algebra (MBA) is an attempt to generalize Boolean algebra. This paper deals with the notion of MBA as introduced by him and is an attempt to give proofs of existence theorems in more detail. Isomorphism of multiple Boolean algebras is defined which is followed by some structure determining theorems.

II. PRELIMINARIES

Zenzo showed that the set of all fuzzy subsets of a set becomes a Multiple Boolean Algebra if the binary operations on it are defined in a suitable manner as follows:

Let p be any integer greater than 1. Multiple Boolean algebra of order p is a set E with p binary operations 0, 1, 2, 3, ...,p-1; p distinguished elements e_0, e_1, e_2, . . e_n and a bijection U : E →E such that the following axioms are satisfied :

For every x, y, z Є E and for every m=0, 1, 2, . . . , p-1,

MBA 1 $x \underline{m} x = x$

MBA 2 $x \underline{m} y = y \underline{m} x$

MBA 3 $(x \underline{m} y) \underline{m} z = x \underline{m} (y \underline{m} z)$

MBA 4 $x \underline{m} e_m = x$

MBA 5 for each $\underline{m}$, there exists a_m Є E such that $x \underline{m} a_m = a_m$

MBA 6 $x \underline{m} (y \underline{m+1} z) = (x m y) \underline{m+1} (x m z)$

MBA 7 $U(x \underline{m} y) = U(x) \underline{m+1} U(y)$

MBA 8 $U^p (x) = x$ i.e. U(U(U(. . . U(U(x)…) = x

MBA 9 $x \underline{m} U(x) \underline{m} U2(x) \underline{m} . . . . . U\underline{p-1}(x)= a_m$

Note 1. Clearly, the a_m is the generalization of zero element, and e_m is generalization of identity element. Axioms 1, 2, 3 show that each operation is idempotent, commutative and associative respectively. MBA4 shows that the element e_m in E is the identity element for the operation $\underline{m}$. The element a_m characterized by MBA5 absorbs every other element in E under the operation $\underline{m}$. It is called as an 'absorbing element'. Axiom 6 requires that each operation be distributive with respect to immediately following one. Axioms MBA7 and MBA8 state that U is an isomorphism carrying (E, $\underline{m}$) over onto (E, $\underline{m+1}$) for every m=0,1,2, . . ., p-2 and (E, $\underline{p-1}$) over onto (E, $\underline{0}$). Zenzo calls this U as 'fundamental isomorphism of the algebra'. Clearly, U is the generalization of complementation operation from Boolean algebra theory and MBA7 is the generalization of 'involution'. Axiom MBA9 generalizes the fact that in ordinary Boolean algebra, the meet and join of an element with its complement are equal to the zero and the identity elements respectively.

Note 2. I think it necessary to add the following axiom

MBA 10 $a_i \neq a_j$ if $i \neq j$

which is not the part of Zenzo's definition, but is required for the infinite case and assumes the uniqueness of 'absorbing' or 'zero' elements for each operation in the infinite case. This point will be illustrated in the papers following this.

Note 3. If E if finite, the existence of absorbing element a_m can be proved using axioms as follows :-

Let E = { x_1, x_2, x_3, … . x_n} being finite.

Then $a_m = x_1 \underline{m} x_2 \underline{m} x_3 . . . . . .\underline{m} x_n$

for, $x_i \underline{m} a_m = x_i \underline{m} (x_1 \underline{m} x_2 . . . . . .\underline{m} x_n)$

$= x_1 \underline{m} x_2 \underline{m} x_3 . . . . . \underline{m} x_n$ (by MBA 1, 2, 3)

$= a_m$ for any x_i Є A

Thus, MBA5 can be omitted when E is finite.

Theorem 2.1 : The identity element e_m and the absorbing element a_m are unique for the operation $\underline{m}$.

Proof : (a) Suppose for an operation $\underline{m}$ there are two identities e_m *and* e_m'. Then, $e_m' = e_m' \underline{m} e_m$ (as e_m is identity for $\underline{m}$)

$= e_m$ (as e_m' is identity for $\underline{m}$)

Hence $e_m' = e_m$

i. e. there is only one identity.

(b) Suppose for an operation m there are two absorbing elements a_m and a'_m.

Then, $a'_m = a'_m$ $\underline{m}$ a_m (taking a'_m as absorbing element)

= a_m (taking a_m as absorbing element)

Theorem 2.2 : $U(e_m) = e_{m+1}$ and $U(a_m) = a_{m+1}$ $\forall\, m = 0, 1, 2 \ldots \ldots \ldots, p-1$

Proof : To prove $U(e_m) = e_{m+1}$, we have to prove that x $\underline{m+1}$ $U(e_m) = x$ $\forall\, x \in E$

Consider, x $\underline{m+1}$ $U(e_m) = U(U^{-1}(x)\ m\ e_m)$ by MBA7

$= U(U^{-1}(x))$ (as e_m is identity element)

= x (hence proof)

Similarly, x $\underline{m+1}$ $U(a_m) = U(U^{-1}(x)\ \underline{m}\ a_m)$

$= U(a_m)$ $\forall\, x \in E$

Hence, $U(a_m) = a_{m+1}$

Remark: It is understood that $\underline{p} = \underline{0}$

Remark: For p=2, the MBA reduces to ordinary Boolean algebra where U(x) is interpreted as complement of x. Theorem 2.1 generalizes the fact that 0 and 1 are unique in ordinary Boolean algebra.

Theorem 2.3: (Existence theorem) For every integer p ≥ 2, there exists a MBA of order p and cardinality p.

Proof : Let I(p) denote the set of first p non-negative integers

i.e. I(p) = {0, 1, 2, 3, . . . p-1}

Define function u : I(p) →I(p) as follows

U(p-1) = 0 and u(j) = j+1 for j = 0,1,2,. . . . ., p-2.

Let the p operations on I(p) be defined by

h $\underline{0}$ k = min(h, k) and h $\underline{m+1}$ k = $u(u^{-1}h\ \underline{m}\ u^{-1} k)$ $\forall$ h, k $\in$ I(p) and m =0 to p-1

Then, clearly, u is a bijection. The first operation, $\underline{0}$ is the 'minimum' operation and hence is clearly associative, commutative and idempotent; and due to the same reason, the biggest integer p-1 in I(p) is identity of $\underline{0}$ and the least 0 (zero) is the absorbing element of it.

Also, by the definition of the binary operations viz.

h $\underline{m+1}$ k = $u(u^{-1}h\ \underline{m}\ u^{-1} k)$,

We get, u(h $\underline{0}$ k) = $u(u^{-1}(uh)\ \underline{0}\ u^{-1}(uk))$

= uh $\underline{1}$ uk

which shows (I(p), $\underline{0}$) is isomorphic with (I(p), $\underline{1}$).

Thus, each operation is isomorphic with $\underline{0}$ and hence is associative, commutative and idempotent and equipped with an identity $e_m = u(e_{m-1})$ and an absorbing element $a_m = u(a_{m-1})$.

We shall prove the distributivity by induction. First, let us prove that $\underline{p-1}$ distributes over $\underline{0}$.

We want, x $\underline{p-1}$ (y $\underline{0}$ z) = (x $\underline{p-1}$ y) $\underline{0}$ (x $\underline{p-1}$ z) $\forall$ x, y, z $\in$ I(p)

i.e. u^{-1} (ux $\underline{0}$ u(y$\underline{0}$z)) = u^{-1} { (ux $\underline{0}$ uy) $\underline{1}$ (ux $\underline{0}$ uz) } . . . . (I)

For clarity, consider, for h, k $\in$ I(p),

h $\underline{p-1}$ k = u^{-1} (uh) $\underline{p-1}$ u^{-1} (uk)

= u^{-1}(uh $\underline{0}$ uk) by def.

Hence, x $\underline{p-1}$ (y $\underline{0}$ z) = u^{-1} (ux $\underline{0}$ u(y $\underline{0}$ z)) and, u^{-1}(h $\underline{1}$ k) = u^{-1}h $\underline{0}$ u^{-1}k by def.

Therefore, u^{-1}{(ux $\underline{0}$ uy) $\underline{1}$ (ux $\underline{0}$ uz)} = u^{-1}(ux $\underline{0}$ uy) $\underline{0}$ u^{-1}(ux $\underline{0}$ uz)

= (x $\underline{p-1}$ y) $\underline{0}$ (x $\underline{p-1}$ z) by def.

Now, after applying u on both sides of eq.(I) we get,

ux $\underline{0}$ u(y$\underline{0}$z) = (ux $\underline{0}$ uy) $\underline{1}$ (ux $\underline{0}$ uz) . . . . . (II)

The different cases in which eq(II) can be proved valid are as follows :

Case 1 : If any one of the x, y and z is equal to p-1 we have,

(i) X = p-1 then (II) becomes,

0 $\underline{0}$ u(y $\underline{0}$ z) = (0 $\underline{0}$ uy) $\underline{1}$ (0 $\underline{0}$ uz)

As 0 is absorbing element of $\underline{0}$, 0 = 0 $\underline{1}$ 0 = 0 which is true.

(ii) y = p-1 then (II) becomes ,

ux $\underline{0}$ u(p-1 $\underline{0}$ z) = [ux $\underline{0}$ u(p-1)] $\underline{1}$ [(ux $\underline{0}$ uz)]

And as p-1 is identity of $\underline{0}$, ux $\underline{0}$ uz = (ux $\underline{0}$ 0) $\underline{1}$ (ux $\underline{0}$ uz)

= 0 $\underline{1}$ (ux $\underline{0}$ uz)

= ux $\underline{0}$ uz

(as u(p-1)=0 is identity of $\underline{1}$)

The same happens in the third case of z=p-1.

Case 2 : Let x, y, z ≠ p-1 and x $\leq$ y. Then, either x $\leq$ z or x $\geq$ z.

(i) Let x $\leq$ z. Again there are sub-cases :-

(a) z $\leq$ y. Then x $\leq$ z $\leq$ y, so that u(x) $\leq$ u(y), as, no one of them is p-1 and definition of u on I(p), eq(II) gives,

ux $\underline{0}$ u(y $\underline{0}$ z) = (ux $\underline{0}$ uy) $\underline{1}$ (ux 0 uz)

i.e. ux $\underline{0}$ u(z) = uz $\underline{1}$ ux

i.e. ux = ux which is true.

(b) z $\geq$ y. Then x $\leq$ y $\leq$ z. Hence we want by (II), ux $\underline{0}$ uy = ux $\underline{1}$ ux

i.e. ux = ux , which is true.

(ii) Let x $\geq$ z. Then, y $\geq$ x $\geq$ z, and eq(II) gives, ux $\underline{0}$ uy = ux $\underline{1}$ ux

i.e. ux = ux which is true.

Case 3 : Let x $\geq$ y where x, y, z ≠ p-1. Here again we get 2 sub-cases

(i) $x \leq z$. Then $z \geq x \geq y$

(ii) $x \geq z$ Then either $z \geq y$ i.e $x \geq z \leq y$ or $z \geq y$ i.e $x \geq y \geq z$.

We see that these are symmetric cases with those we have proved above. Hence we drop the proofs.

Thus, first step of induction is proved.

In the second step, let us assume that m distributes over $\underline{m+1}$. Since the set of operations is cyclic and finite, we show that $\underline{m-1}$ distributes over $\underline{m}$.

$x \ \underline{m-1}\ (y \ \underline{m}\ k) = u^{-1} (ux \ \underline{m}\ u(y \ \underline{m}\ k))$ by def.

$= u^{-1} (ux \ \underline{m}\ (uy \ \underline{m+1}\ uk))$ (proved in the following remark)

$= u^{-1} ((ux \ \underline{m}\ uy) \ \underline{m+1}\ (ux \ \underline{m}\ uk))$ by induction

$= u^{-1} (u(x \ \underline{m}\ y) \ \underline{m+1}\ u(x \ \underline{m-1}\ k))$

$= u^{-1}(u [(x \ \underline{m}\ y) \ \underline{m}\ (x \ \underline{m-1}\ k)])$

$= (x \ \underline{m-1}\ y) \ m (x \ \underline{m-1}\ k)$

Thus, we proved MBA1 to MBA6.

Remark : As $x \ \underline{m+1}\ y = u(u^{-1} x \ \underline{m}\ u^{-1}y)$, $x \ \underline{m}\ y = u^{-1}(ux \ \underline{m+1}\ uy)$

Which proves $u(x \ \underline{m}\ y) = ux \ \underline{m+1}\ uy$

This proves MBA7.

For MBA8, we observe that $u^p(x)$ shifts the integer x through p places; and u(p-1) being 0, we get (as I(p) contains p integers), $u^p(x) = x$.

Now, by definition of u, for any $x \in I(p)$, the elements x, u(x), u(u(x)), ..., $u^{p-1}(x)$ are all the elements of I(p). Hence in view of Note 2 above,

$x \ \underline{m}\ u(x) \ \underline{m}\ u(u(x)) \ \underline{m}\ . . . . . \underline{m}\ u^{p-1}(x) = a_m$

This proves MBA9.

The theorem assures existence of MBA having as many binary operations as it has elements. Note that for p=2, I(2) coincides with B_2. Zenzo calls these by the name 'Basic MBA'. Really, these are 'basic' as we shall see in the next theorem, that they can be used to generate multiple Boolean algebras of cardinality p^n. Before that I give here I(4), that is basic MBA of order 4 and cardinality 4.

Example 1 : The base set is I(4) = { 0, 1, 2, 3}. The operations $\underline{0}$, $\underline{1}$, $\underline{2}$, $\underline{3}$ are given by the following tables.

$\underline{0}$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$\underline{1}$	0	1	2	3
0	0	1	2	3
1	1	1	1	1
2	2	1	2	2
3	3	1	2	3

$\underline{2}$	0	1	2	3
0	0	0	2	3
1	0	1	2	3
2	2	2	2	2
3	3	3	2	3

$\underline{3}$	0	1	2	3
0	0	0	0	3
1	0	1	1	3
2	0	1	2	3
3	3	3	3	3

In this, the identity elements are $e_0 = 3$, $e_1 = 0$, $e_2 = 1$, $e_3 = 2$

The absorbing elements are $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$.

The fundamental isomorphism is

u(0) =1, u(1) =2, u(2) = 3, u(3) = 0

That is, $u \equiv (1, 2, 3, 0)$.

III. POWER SET OF MULTIPLE BOOLEAN ALGEBRAS

Theorem 3.1 :Let I(p) be the basic MBA of order p. Let A be any set and E be the set of all functions $f: A \to I(p)$

Define p binary operations $\overline{0}, \overline{1}, \overline{2}, ... \overline{p-1}$ on E as follows :

For m=0, 1, 2, 3. . , p-1 and f, g, h $\in$ E such that

$f \ \overline{m}\ g = h$ iff $h(x) = f(x) \ \underline{m}\ g(x) \ \forall x \in A$

Let $v : E \to E$ be defined by

$v(f) = k$ iff $k(x)= u(f(x)) \ \forall x \in A$; $f, k \in E$.

Then, with these p operations and unary operation v on E, E is a multiple Boolean algebra of order p in which the identities and absorbing elements are precisely defined.

Proof : (A) Due to the corresponding properties of the operations in I(p) and by definition of $\overline{m}$ on E, each binary operation $\overline{m}$ on E is associative, commutative and idempotent.

The constant function,

$f_m : A \to I(p)$ defined by $f_m(x) = e_m \ \forall x \in A$

is an element of E and acts as identity element for the operation $\overline{m}$. Indeed, for every $x \in A$ and $g \in E$

$(g \ \overline{m}\ f_m)(x) = g(x) \ \underline{m}\ f_m(x)$ by def.

$= g(x) \ \underline{m}\ e_m$

$= g(x)$

Thus, $g \ \overline{m}\ f_m = g \quad \forall g \in E$

B) Distributivity: Consider, for $x \in A$ and f, g, h $\in$ E

$[f \ \overline{m}\ (g \ \overline{m+1}\ h)] (x)$

$= f(x) \ \underline{m}\ [(g \ \overline{m+1}\ h) (x)]$

$= f(x) \ \underline{m}\ [g(x) \ \overline{m+1}\ h(x)]$

$= [f(x) \ \underline{m}\ g(x)] \ \underline{m+1}\ [f(x) \ \underline{m}\ h(x)]$

$= [(f \ \overline{m}\ g) x] \ \underline{m+1}\ [(f \ \overline{m}\ h) x]$

$= [(f m g) \ \underline{m+1}\ (f m h) (x)$

$\therefore [f \ \overline{m}\ (g \ \overline{m+1}\ h)] \equiv [(f \ \overline{m}\ g) \ \overline{m+1}\ (f \ \overline{m}\ h)]$

C) Since u is well defined, v is **well defined** function.

Claim : v is one-one i.e. injective

Proof : Let f, g $\in$ E such that $f \not\equiv g$.

Then, $f(x) \neq g(x)$ for some $x \in A$

$\Rightarrow (vf)\ x = u(f(x)) \neq u(g(x) = (vg)\ x$ for that x, since u is one-one.

$\Rightarrow$ vf $\not\equiv$ vg
Thus, v is injective.

Claim : v is surjective.

Proof : Let g∈ E. Define f: A → I(p) by f(x) = u^{-1}(g(x)),
Then, (vf) (x) = u(f(x)) = g(x) ∀ c ∈ A.
Hence v (f) = g.
Thus for every g ∈ E, ∃ f ∈ E such that v(f) = g i.e. v is surjective.

Claim : v(f $\overline{m}$ g)) = v(f) $\overline{m+1}$ v(g)

Proof : For x ∈ A, v(f $\overline{m}$ g)) (x) = u[(f $\overline{m}$ g)(x)] = u (f(x) $\underline{m}$ g(x))
= u(f(x)) $\underline{m+1}$ u(g(x))
= vf(x) $\underline{m+1}$ vg(x)
= (vf $\overline{m+1}$ vg) (x)
∴v(f $\overline{m}$ g)) = v(f) $\overline{m+1}$ v(g)
Thus MBA7 is satisfied

D) Claim : $v^p(f) = f$ ∀ f ∈ E
Proof : $(v^p(f))x = v(v^{p-1}f)x$
= $u(v^{p-1}f(x))$
= $u(u(v^{p-2}f(x)))$
= ⋮
= $u^p(v^0f(x))$
= f(x) ∀ x ∈ A Hence proof.

Next, the absorbing element for the operation $\underline{m}$ on E is the function L_m : A → I(p) defined by $L_m(x) = a_m$ ∀ x ∈ A . Indeed if f is any other element of E,

$$(f\,\overline{m}\,L_m)(x) = f(x)\,\underline{m}\,L_m(x) = f(x)\,\underline{m}\,a_m = a_m \text{ for all } x \in A.$$

That is,

$$f\,\overline{m}\,L_m = L_m \text{ for all } f \in E$$

Thus, existence of absorbing element is proved. Moreover, since absorbing elements in I(p) are distinct, those in E are distinct.

i.e. $L_i \neq L_j$ if $i \neq j$ $0 \leq i,j \leq p-1$
This proves MBA 6 and MBA10.

Now, for MBA9, consider,

$$\{[f\,\underline{m}\,v(f)\,\underline{m}\,v^2(f)\,\underline{m}\ldots\underline{m}\,v^{p-1}(f)]\,\underline{m}\,g\}(x) \quad \text{where } x \in A;\ f,g \subset E$$

$$= [f\,\overline{m}\,v(f)\,\overline{m}\,v^2(f)\,\overline{m}\ldots\overline{m}\,v^{p-1}(f)](x)\ \underline{m}\ g(x)$$

$$= [f(x)\,\underline{m}\,u(f(x))\,\underline{m}\,u^2(f(x))\,\underline{m}\ldots\underline{m}\,u^{p-1}(f(x))]\ \underline{m}\ g(x)$$

= a_m $\underline{m}$ g(x)
= a_m
Hence by definition of absorbing element L_m in E,

$$[f\,\overline{m}\,v(f)\,\overline{m}\,v^2(f)\,\overline{m}\ldots\overline{m}\,v^{p-1}(f)]\ \overline{m}\ g = L_m \quad \forall\, g \in E$$

Hence by uniqueness of L_m ,

$$f\,\overline{m}\,v(f)\,\overline{m}\,v^2(f)\,\overline{m}\ldots\overline{m}\,v^{p-1}(f) = L_m \text{ for any } f \in E$$

This completes the proof.

Remark 1: The above theorem gives us Infinite MBA. For that we have to just take the set A to be infinite. It generalizes the fact that in ordinary Boolean algebra theory, the power set of any set x is also a Boolean algebra. Indeed, if we take E to be set of all functions
f: X→ I(2) i.e f : X→ { 0, 1 }
and define operations ∩ and ∪ on E by
f ∩ g = h iff h(x) = f(x) $\underline{0}$ g(x) ∀ x ∈ E
f ∪ g = k iff k(x) = f(x) $\underline{1}$ g(x) ∀ x ∈ E
then E becomes set of all characteristic functions of X, i.e. the set of all subsets of X.

Remark 2: If we take the set A to be finite of cardinality n, then E becomes a MBA of order p and cardinality p^n. Thus, for a given pair of integers $p \geq 2$ and $n \geq 1$, there exists a MBA of order p and cardinality p^n. We shall call this 'Power set of multiple Boolean algebra' in the absence of better words.

We illustrate the above theorem by an example.

Example 2: Let E be the set of all functions f from A = {x, y} to I(3). Let us denote the $3^2 = 9$ functions by integers ranging from 0 to 8 as follows :
Ordered pair (a, b), a, b ∈ I(3) denotes the function which maps x to a and y to b. For example, we denote 7≡(2,1), which means the function which maps x to 2 and y to 1 is denoted by integer 7. Thus, the functions are denoted by :
0 ≡(0, 0), 1 ≡(0, 1), 2 ≡(0, 2), 3 ≡(1, 0), 4 ≡(1, 1), 5 ≡(1, 2), 6 ≡(2, 0), 7 ≡ (2, 1), 8 ≡(2, 2).
Then, the operations $\overline{0}, \overline{1}, \overline{2}$ defined according to above theorem can be given by the following tables.

$\bar{0}$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	1	0	1	1	0	1	1
2	0	1	2	0	1	2	0	1	2
3	0	0	0	3	3	3	3	3	3
4	0	1	1	3	4	4	3	4	4
5	0	1	2	3	4	5	3	4	5
6	0	0	0	3	3	3	6	6	6
7	0	1	1	3	4	4	6	7	7
8	0	1	2	3	4	5	6	7	8

$\bar{1}$	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	1	1	4	4	4	7	7	7
2	2	1	2	5	4	5	8	7	8
3	3	4	5	3	4	5	3	4	5
4	4	4	4	4	4	4	4	4	4
5	5	4	5	5	4	5	5	4	5
6	6	7	8	3	4	5	6	7	8
7	7	7	7	4	4	4	7	7	7
8	8	7	8	5	4	5	8	7	8

$\bar{2}$	0	1	2	3	4	5	6	7	8
0	0	0	2	0	0	2	6	6	8
1	0	1	2	0	1	2	6	7	8
2	2	2	2	2	2	2	8	8	8
3	0	0	2	3	3	5	6	6	8
4	0	1	2	3	4	5	6	7	8
5	2	2	2	5	5	5	8	8	8
6	6	6	8	6	6	8	6	6	8
7	6	7	8	6	7	8	6	7	8
8	8	8	8	8	8	8	8	8	8

In this example, the identities are $e_{\bar{0}} = 8,\ e_{\bar{1}} = 0,\ e_{\bar{2}} = 4$
And the absorbing elements are $a_{\bar{0}} = 0,\ a_{\bar{1}} = 4,\ a_{\bar{2}} = 8$.

The fundamental isomorphism V : E → E is defined by
V(0) = 4, V(1)=5, V(2) = 3, V(3)=7, V(4) = 8, V(5)=6, V(6) = 1, V(7)=2
And V(8) = 0.
In short, V ≡ (4, 5, 3, 7, 8, 6, 1, 2, 0).

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