# Double Geodetic Domination Number of A Graph

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***Abstract****-A subset S of vertices in a graph G is a called a double geodetic dominating set if S is both a double geodetic set and a (standard) dominating set. The double geodetic domination number γ_DG (G) is the minimum cardinality of a double geodetic dominating set. Any double geodetic domination of cardinality γ_DG (G) is called γ_DG-set of G. In this paper, we study double geodetic domination on graphs.*

***Keywords***-Domination, geodesic, geodetic, double geodetic, double geodetic dominating set, double geodetic domination number.
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## I. INTRODUCTION

We consider finite graphs without loops and multiple edges. For any graph G the set of vertices is denoted by *V(G)* and the edge set by *E(G).* We define the order of *G* by $n = n(G) = |V(G)|$ and the size by $m = m(G) = |E(G)|$. For a vertex$v \in V(G)$, the *open neighborhood N(v)* is the set of all vertices adjacent to *v*, and *N[v] = N(v)∪{v}* is the *closed neighborhood* of *v*. The *degree d(v)* of a vertex *v* is defined by *d(v) = |N(v)|*. The *minimum* and *maximum degrees of a graph G are* denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively.

For $X \subseteq V(G)$let *G[X]* the sub graph of *G* induced by *X*, *N(X)* = $\cup_{x\in X}$*N(x)* and *N[X]* =$\cup_{x\in X}$*N(x).* If *G* is a connected graph, then the distance *d(x, y)* is the length of a shortest $x - y$ path in *G*. The diameter $diam(G)$ of a connected graph is defined by *diam(G)* = $max_{x,y\in V(G)} d(x,y)$. An $x - y$ path of length *d(x, y)* is called an $x - y$ *geodesic*. A vertex *v* is said to lie on an $x - y$ geodesic *P* if *v* is an internal vertex of *P*. The closed interval *I[x, y]* consists of *x*, *y* and all vertices lying on some $x - y$ geodesic of *G*, while for $S \subseteq V(G)$, *I[S]* = $\cup_{x,y\in S} I[x,y]$. If *G* is a connected graph, then a set *S* of vertices is a *geodetic set* if *I[S] = V(G).* The minimum cardinality of a geodetic set is the *geodetic number* of *G*, and is denoted by *g(G).* The geodetic number of a disconnected graph is the sum of the geodetic numbers of its components. A geodetic set of cardinality $g(G)$ is called a $g(G)$ *−set*. For references on geodetic sets see [1, 2, 3, 4, 6 ].

Let *G* be a connected graph with at least two vertices. A set *S* of vertices of *G* is called a *double geodetic set* of *G* if for each pair of vertices *x*, *y* in *G* there exist vertices *u*, *v* ∈*S* such that x, y ∈I[u, v]. The *double geodeticnumber* $dg(G)$ of G is the minimum cardinality of a double geodetic set. Any double geodetic set of cardinality $dg(G)$ is called $dg(G)$*-set* of G. [8]

A vertex in a graphdominates itself and its neighbors. A set of vertices *S* in a graph *G* is a *dominating set* if $N[S] = V(G)$. The *domination number* $\gamma(G)$ of *G* is the minimum cardinality of a dominating set of *G*. The domination number was introduced in [6].

It is easily seen that a dominating set is not in general a double geodetic set in a graph G. Also the converse is not valid in general. This has motivated us to study the new domination conception of double geodetic domination. We investigate those subsets of vertices of a graph that are both a double geodetic set and a dominating set. We call these sets double geodetic dominating sets. We call the minimum cardinality of a double geodetic dominating set of G, the double geodetic domination number of G. It is easily seen that every extreme vertex belongs to every double geodetic dominating set.

The following theorems are used in the sequel.

**1.1 Theorem:[6]** The domination number of some standard graphs are given as follow.

1. $\gamma(P_p) = \left\lceil \frac{p}{3} \right\rceil, p \geq 2$.
2. $\gamma(C_p) = \left\lceil \frac{p}{3} \right\rceil, p \geq 3$, where $\lceil x \rceil$denotes the smallest positive integer greater than or equal to x
3. $\gamma(K_p) = \gamma(W_p) = \gamma(K_{1,n}) = 1$.
4. $\gamma(K_{n,m}) = 2 \;\; if \; m,n \geq 2$.

**1.2 Remark:[6]**

1) A dominating set D of G is a minimal dominating set of G if and only if for every $v \in D$, there exists at least one vertex $w \in V - (D - \{v\})$ which is not dominated by $D - \{v\}$.
2) A dominating set D of G is a minimal dominating set of G if and only if for every $v \in D$, there exists at least one vertex $w \in V - (D - \{v\})$ such that $N[w] \cap D = \{v\}$.

**1.3 Theorem:[8]** For any graph G of order p, $2 \leq g(G) \leq dg(G) \leq p$.

**1.4 Theorem:[8]** Each extreme vertex of a connected graph G belongs to every double geodetic set of G. In particular, if the set of all end vertices of G is a double geodetic set, then it is the unique $dg - set$ of G.

**1.5 Theorem:[8]** For the complete graph $K_n (n \geq 2)$, we have $dg(K_n) = n$..

**1.6 Definition:** Two vertices u and v of a graphs G are said to be antipodal vertices G if $d(u, v) = diam\ G$. Every graph has at least two antipodal vertices.

## II. DOUBLE GEODETIC DOMINATION NUMBER OF GRAPHS

**2.1 Definition:** Let $G = (V, E)$ be any connected graph with at least two vertices. A double geodetic dominating set of G is a subset S of $V(G)$ which is both dominating and double geodetic set of G.

A double geodetic dominating set S is said to be a minimal double geodetic dominating set of G if no proper subset of S is a double geodetic dominating set of G. A double geodetic dominating set S is a said to be a minimum double geodetic dominating set of G if there exists no double geodetic dominating set $S^1$ such that $| S^1 | < | S |$. The cardinality of a minimum double geodetic dominating set of G is called the double geodetic domination number of G. It is denoted by $\gamma_{DG}(G)$. Any double geodetic dominating set S of G of cardinality $\gamma_{DG}$ is called a $\gamma_{DG} -$ set of G.

**2.2 Remark:** Let G be a connected graph with $p \geq 2$ vertices. Then, $\gamma(G) \leq \gamma_{DG}(G)$. Strict inequality is also true in the above relation. For example, consider $P_{15}$. Let $V(P_{15}) = \{ v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$. $\gamma(P_{15}) = 5$, since $\{v_2, v_5, v_8, v_{11}, v_{14}\}$ is a $\gamma - set$. $\gamma_{DG}(P_{15}) = 6$, since $\{v_1, v_4, v_7, v_{10}, v_{13}, v_{15}\}$ is a $\gamma_{DG} - set\ of\ P_{15}$.

**2.3 Example** Considering the graph G in figure 2.3(a),

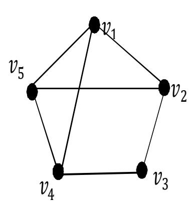


Figure 2.3(a)

$\{v_2, v_4\}$ is the unique dg-set of G. So dg(G) = 2
$\{v_2, v_4\}$ is also the unique $\gamma_{DG} - set$ of G. Therefore, $\gamma_{DG}(G) = 2$
Also $\gamma(G) = 2$, though G has more than one $\gamma - set$.
In this example, dg(G) = $\gamma_{DG}(G) = \gamma(G)$.

**2.4 Remark:**

In general, $\gamma_{DG}(G)$, dg(G) and $\gamma(G)$, all need not be equal.
For example consider $P_8$. $\gamma_{DG}(P_8) = \{ v_1, v_4, v_6, v_8\} = 4$, dg$(P_8) = \{ v_1, v_8\} = 2$ and $\gamma(P_8) = \{v_2, v_5, v_8\} = 3$.
Further, $\gamma_{DG}(P_7) = 3 = \gamma(P_7)$, But dg$(P_7) = 2$.
Also, $\gamma_{DG}(P_3) = dg(P_3) = 2$, whereas, $\gamma(P_3) = 1$.

**2.5 Remark:** Let G = ( V,E ) be any connected graph with at least two vertices. Then,

1. $\gamma_{DG}(G) \geq dg(G)$ and $\gamma_{DG}(G) \geq \gamma(G)$.
2. Every double geodetic dominating set of G contains all pendant vertices of G.
3. If G is a graph with at least one pendent vertex, then for every double geodetic dominating Set D of G, V-D is not a double geodetic dominating set of G.
4. Every super set of a double geodetic dominating set of G is a double geodetic dominating setof G.

**2.6 Proposition:** For a star graph G, then $\gamma_{DG}(G) = p - 1$.

**Proof:** Let G = $K_{1,n}$ with $V(K_{1,n}) = \{v, v_i : 1 \leq i \leq n\}$ and $E(K_{1,n}) = \{vv_i : 1 \leq i \leq n\}$. Let S be a minimum double geodetic dominating set of $K_{1,n}$. By Remark 2.5$\{v_1, v_2, v_3, \ldots\ldots . v_n\} \subseteq S$. Since $\{v_1, v_2, v_3, \ldots\ldots . v_n\}$ itself is a double geodetic dominating set of $K_{1,n}$ , S = $\{v_1, v_2, v_3, \ldots\ldots, v_n\}$. Therefore, $\gamma_{DG}(K_{1,n}) = n = p - 1$.

**2.7 Proposition:** If G is a bi-star graph G, then $\gamma_{DG}(G) = p - 2$.

**Proof:** Let G = $B(r, s)$ where $r, s \geq 1$. Suppose $V(B(r, s)) = \{u, v, u_i, v_i : 1 \leq i \leq r\ and\ 1 \leq j \leq s\}$ and $E(B(r, s)) = \{uv, uu_i, vv_i : 1 \leq i \leq r\ and\ 1 \leq j \leq s\}$. Let S be a minimum double geodetic dominating set of $B(r, s)$. By Remark 2.5.$\{u_1, u_2, \ldots\ldots . u_r, v_1, v_2, \ldots\ldots . v_s\} \subseteq S$. Since $\{u_1, u_2, \ldots\ldots . u_r, v_1, v_2, \ldots\ldots . v_s\}$ is itself a double geodetic dominating set of G,$S = \{ u_1, u_2, \ldots\ldots . u_r, v_1, v_2, \ldots\ldots . v_s\}$. Therefore, $\gamma_{DG}(B(r, s)) = p - 1$.

**2.8. Remark.**

$$\left\lceil \frac{n-4}{3} \right\rceil + 2 = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & if\ n \equiv 1\ (mod\ 3) \\ \left\lceil \frac{n}{3} \right\rceil + 1 & otherwise \end{cases}$$

**2.9.Theorem** $\gamma_{DG}(P_n) = \begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 & if\ n \geq 5 \\ 2 & if\ n = 2,3\ or\ 4 \end{cases}$

**Proof.** Let $P_n = \{v_1, v_2, v_3, \dots \dots . v_n\}$. If n = 2,3,or 4, then $\{v_1, v_n\}$ is a minimum double geodetic dominating set of $P_n$.therefore, $\gamma_{DG}(P_2) = \gamma_{DG}(P_3) = \gamma_{DG}(P_4) = 2$ . Let $n \geq 5$.We observe that every double geodetic dominating set of $P_n$ is a dominating set containing the end vertices of $P_n$. Let $D_1$be a minimum dominating set containing $v_1, v_n$. Therefore, $|D_1| \leq |S|$. As $D_1$ is also a double geodetic dominating set of G,$|S| \leq |D_1|$. So, we have, $\gamma_{DG}(P_n) = |S| = |D_1|$. Let D be a minimum dominating set of $P_n$. Then,

$$| D_1| = \begin{cases} |D| = \left\lceil \frac{n}{3} \right\rceil & if\ n \equiv 1\ (mod\ 3) \\ |D| + 1 = \left\lceil \frac{n}{3} \right\rceil + 1 & otherwise \end{cases}$$

$= \left\lceil \frac{n-4}{3} \right\rceil + 2$. (By Remark 2.8)

Therefore,$\gamma_{DG}(P_n) = \left\lceil \frac{n-4}{3} \right\rceil + 2$.

**2.10. Theorem:** $2 \leq \gamma_{DG}(G) \leq p$.

**Proof:** By Remark 2.5 and by theorem 1.3, $\gamma_{DG}(G) \geq 2$ . Further, any double geodetic dominating set is a subset of V(G) implies $\gamma_{DG}(G) \leq p$.

In the above proposition, upper bound is sharp, as $\gamma_{DG}(K_p) = p$ and the lower bound is sharp (see example 2.3)

**2.11. Theorem:** Let G be any graph with k support vertices and $l$ end vertices. Then,

$l \leq \gamma_{DG}(G) \leq p - k$.

**Proof:** LetL and K denote the set of all end and support vertices of G respectively and$|L| = l$ ; $|K| = k$. Clearly, $l \geq k$. By Remark 2.5, L is a subset of every double geodetic dominating set of G. So, $\gamma_{DG}(G) \geq l$. Further every vertex of K lies in a geodesic joining two vertices of L as well as dominated by the vertices of L. Therefore, it is clear that $V - K$ is a double geodetic dominating set of G and so $\gamma_{DG}(G) \leq |V - K| = |V| - |K| = p - k$. Hence the proof.

**2.12. Corollary:** Let T be any tree with k support vertices and $l$ end vertices such that$l + k = p$. Then, $\gamma_{DG}(G) = p - k$.

The following example shows even if $l + k < p$, $\gamma_{DG}(G) = p - k$.

**2.13. Example:** consider the graph G in figure 2.13(a)

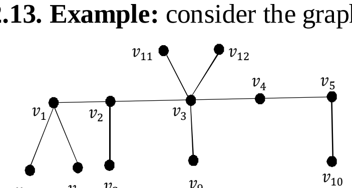


Figure 2.13(a)

Clearly, $S = \{v_4, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}\}$ is a$\gamma_{DG}(G) = 8 = |V - K|$.

**2.14. Theorem:** Let G be a connected graph of order $p(\geq 2)$ and diameter $d$. Suppose $\delta \geq 3$. Then, $\gamma_{DG}(G) \leq p - d + 1$.

**Proof:** Let $u, v \in V(G)$.As diam(G)=d, There exists a shortest path $P$: $(u = v_0, v_1, \dots \dots, v_d = v)$ of length d in G. If S=$\{v_1, v_2, \dots \dots, v_{d-1}\}$, then $V - S$ is a double geodetic set of G. since $\delta \geq 3$, each vertex of S is adjacent to atleast one vertex of $V - S$ is a double geodetic dominating set of G and so $\gamma_{DG}(G) \leq |V - S| = p - (d - 1) = p - d + 1$.

**2.15 Observation:** Let G be connected graph with $p(\geq 2)$ vertices. Then

1. A minimal double geodetic set of G which is also a dominating set of G is a minimal double geodetic dominating set of G.
2. A minimum double geodetic set (or dg-set) of G which is also a dominating set of G is a minimum double geodetic dominating set of G. ( $or\ \ \gamma_{DG} -$ set) of G.
3. Any minimal dominating set of G which is also a double geodetic set of G is a minimal double geodetic dominating set of G.
4. Any minimum dominating set (or $\gamma$-set) of G which is also a double geodetic set of G is a minimum double geodetic dominating set of G. ( $or\ \ \gamma_{DG} -$ set) of G.

## III. FURTHER RESULTS ON DOUBLE GEODETIC DOMINATION NUMBER OF GRAPHS

**3.1. Theorem:** Let G = (V,E) be any graph. Suppose S is a proper subset of V(G) such that the sub graph of G induced by S is complete, then S is not a double geodetic dominating set of G.

**Proof:** Let $v \in V - S$. Since the sub graph of G induced by S is complete, d(x,y)=1 for every $x, y \in S$. Therefore, v does not lie on any double geodesic joining any pair of vertices of S. Hence, S is not a double geodetic dominating set of G.

**3.2. Theorem:** Let G= (V, E) be a graph. Then, every double geodetic dominating set of G contains all the extreme vertices of G.

**Proof:** Let S be the set of all extreme vertices of G. By 1.4, every double geodetic set of G contains S. As every double geodetic dominating set of G is a double geodetic set of G, every double geodetic dominating set of G contains S.

**3.3. Theorem:** Let G = (V, E) be any graph. If S, the set of all extreme vertices of G, is a double geodetic dominating set of G, then S is the unique minimum double geodetic dominating set of G.

**Proof:** Suppose S is a double geodetic dominating set of G. Then, $\gamma_{DG}(G) \leq |S|$.By Theorem 3.2, $\gamma_{DG}(G) \geq |S|$. Therefore, $\gamma_{DG}(G) = |S|$ and so S is a minimum double geodetic dominating set of G. Again by Theorem 3.2, every minimum double geodetic dominating set of G contains S and so S is the unique minimum double geodetic dominating set of G.

**3.4. Remark:** If S, The set of all end vertices of G, is a double geodetic dominating set of G, then S is the unique minimum double geodetic dominating set of G.

**3.5. Theorem:** For every positive integer $k \geq 2$, there exists a graph G with $\gamma_{DG}(G) = k$.

**Proof:** Let $k \geq 2$. Consider the complete bipartite graph$K_{1,k}$. Let $S = V(K_{1,k}) - \{v\}$ where $v$ is the central vertex. Obviously, S is a double geodetic dominating set of G. Therefore, by Remark 3.4, S is the unique minimum double geodetic dominating set of $K_{1,k}$ and so $\gamma_{DG}(K_{1,k}) = k$.

**3.6. Theorem:** For every pair $k, p$ of integer such that $2 \leq k \leq p$, there exists a connected graph G of order p such that $\gamma_{DG}(G) = k$.

**Proof:** As $\gamma_{DG}(K_p) = p$, the result is true when $k = p$. For $k = 2$, the result is true when $G \cong P_3$ is a graph as in figure 3.6(a) . Here $\{u, v\}$ is a minimum double geodetic dominating set of G and so $\gamma_{DG}(G) = 2$.

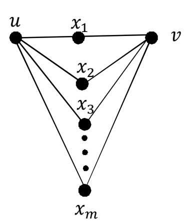



**Figure 3.6(a)**

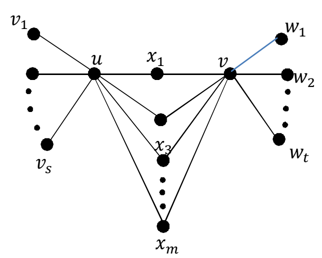



**Figure 3.6(b)**

Let $2 < k < p$. Consider the graph as in figure 3.6(b).

Where $m = p - (k + 1)$.Here, u and v are vertices of G such that there are exactly m paths of length 2 connecting u and v. Let $x_1, x_2, x_3, \dots\dots..x_m$ be the internal vertices of these paths. Let s and t be positive integers such that $1 \leq s, t \leq k - 2$ and $s + t = k - 1$. Attach s end vertices $v_1, v_2, v_3, \dots\dots..v_s$ to u and t end vertices $w_1, w_2, w_3, \dots\dots..w_t$ to v. Then, number of vertices of $G = s + t + 2 + m = k - 1 + 2 + p - k - 1 = p$.Further, $\{v_1, v_2, v_3, \dots\dots, v_s, w_1, w_2, w_3, \dots\dots..w_t, u\}$ is a minimum double geodetic dominating set of G and so $\gamma_{DG}(G) = k$.

## IV. GRAPHS WITH PRESCRIBED DOUBLE GEODETIC DOMINATION NUMBER

**4.1 Proposition:** Let G= (V,E) be a connected graph on p vertices. Then, $\gamma_{DG}(G) = p$ if and only if $dg(G) = p$.

**Proof:** Let $\gamma_{DG}(G) = p$. Assume that $dg(G) \neq p$. Then, there exists at least two vertices u and v such that $d(u, v) \geq 2$. Let $u'$ *and* $v'$ (not necessarily distinct) be the vertices adjacent to $u$ and $v$ respectively in a $u - v$ double geodesic. This implies $V - \{u', v'\}$ is a double geodetic dominating set of G. It contradicts our hypothesis that $\gamma_{DG}(G) = p$. Thus, $dg(G) = p$. The converse follows from Remark 2.5.

**4.2 Corollary:** Let G= (V,E) be a connected graph on p vertices. Then, $\gamma_{DG}(G) = p$ if and only if G is complete.

**Proof:** The result follows form Theorem 1.5 and Proposition 4.1.

**4.3 Theorem:** Let G= (V,E) be a connected graph on p vertices. If $dg(G) = p - 1$, then $\gamma_{DG}(G) = p - 1$.

**Proof:** Suppose $dg(G) = p - 1$. Then, by Remark 2.5 $\gamma_{DG}(G) = p\ (or)\ p - 1$. Therefore, by using Proposition 4.1, $\gamma_{DG}(G) = p - 1$.

**4.4 Theorem:** If $dg(G) = p - 2$, then $\gamma_{DG}(G) = p - 2$.

**Proof:** Let S be a minimum double geodetic set of G. As $dg(G) = p - 2, |V - S| = 2$. Let $V - S = \{u, v\}$. Since S is double geodetic, there exists $x, y \in S$ such that an $x - y$ geodesic P of length at most 3 contains u. So, u is adjacent to either x or y. Similarly, v is adjacent to at least one vertex of S. Therefore, S is also a dominating set G. By Observation 2.15, S is a minimum double geodetic dominating set of G and hence $\gamma_{DG}(G) = p - 2$.

**4.5 Remark:** Converse of the above lemma is not true. For example, $dg(G) = 2 = p - 3$ and $\gamma_{DG}(G) = 3 = p - 2$

**4.6.Theorem:**If$G_1$ *and* $G_2$ *are* $(p_1, q_1)$ *and* $(p_2, q_2)$ complete graphs respectively, then $\gamma_{DG}(G_1 + G_2) = p_1 + p_2 = \gamma_{DG}(G_1) + \gamma_{DG}(G_2)$.

**Proof:** Since $G_1$ *and* $G_2$ *are* complete graphs, $G_1 + G_2$ is a complete graph with $p_1 + p_2$ vertices. Hence, by Corollary 4.2 $\gamma_{DG}(G_1 + G_2) = p_1 + p_2 = \gamma_{DG}(G_1) + \gamma_{DG}(G_2)$.

**4.7 Theorem:** Let G be a connected graph on $p(\geq 2)$ vertices. Then, $\gamma_{DG}(G) + \chi(G) = 2p$ if and only if G is complete.

**Proof:** If $\gamma_{DG}(G) + \chi(G) = 2p$, then $\gamma_{DG}(G) = p$ *and* $\chi(G) = p$. Hence, by corollary 4.2, G is complete. Converse is obvious.

## V. RESULTS CONNECTING DOMINATION, DIAMETER, DOUBLE GEODETIC AND DOUBLE GEODETIC DOMINATION NUMBER OF A GRAPHS

**5.1 Lemma:** Given two positive integer a and b with b > a , there exists a graph G such that $\gamma(G) = a$ and $\gamma_{DG}(G) = dg(G) = b$.

**Proof:** Let a=1 and b>a. The star graph $K_{1,b}$ satisfies the required condition.

Let $a = 2$ and $b > a$. Let s and t be two positive integers such that s+ t = b. Considering G in figure 5.1(a)

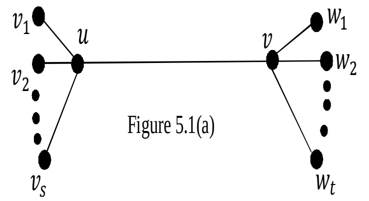


Figure 5.1(a)

$\{u, v\}$ is a minimum dominating set of G and so $\gamma(G) = 2$. Further, $\{v_1, v_2, \dots\dots, v_s, w_1, w_2, \dots, w_t\}$ is a minimum double geodetic and minimum double geodetic dominating set of G and hence $\gamma_{DG}(G) = dg(G) = s + t = b$.

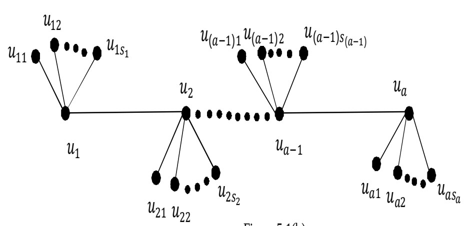


Figure 5.1(b)

Let a>2 and b>a. Let $P_a: (u_1, u_2, \dots\dots, u_a)$. Choose positive integer $s_1, s_2, \dots\dots, s_a$ such that $s_1 + s_2 + \cdots + s_a = b$. Join $s_1, s_2, \dots\dots, s_a$ pendant vertices $u_1, u_2, \dots\dots, u_a$ of $P_a$. The resulting graph appears as in figure 5.1(b) . Clearly, $\{u_1, u_2, \dots\dots, u_a\}$ is a minimum dominating set of G and so $\gamma(G) = a$ and

$\{u_{11}, u_{12}, \dots\dots, u_{1s_1}, u_{21}, u_{22}, \dots\dots, u_{2s_2}, \dots\dots, u_{a1}, u_{a2}, \dots\dots, u_{as_a}\}$ is a $dg-set$ and $\gamma_{DG}-set$ of G and $\gamma_{DG}(G) = dg(G) = b$.

**5.2 Theorem:** If $diam(G) = 1, 2$ or $3$, then $\gamma_{DG}(G) = dg(G)$.

**Proof:** If $diam(G) = 1$, then $G \cong K_p$ and $\gamma_{DG}(G) = p = dg(G)$. Suppose $diam(G) = 2$ or $3$, then every double geodetic set of G is also a dominating set of G. Therefore, every minimum double geodetic set of G is also a minimum double geodetic dominating set of G and so $\gamma_{DG}(G) \le dg(G)$. Hence, by Remark 2.5, $\gamma_{DG}(G) = dg(G)$.

**5.3 Remark:** Converse of the above theorem need not be true since any caterpillar T constructed form a path of length n>3 in which every vertex is either a support vertex or a pendant vertex has diameter greater than 3 and $\gamma_{DG}(T) = dg(T)$.

**5.4 Theorem:** Let G be a non-complete connected graph with $p\ (\ge 4)$ vertices. If $diam\ G \ge 3$, then $\gamma_{DG}(G) \le p - 2$.

**Proof:** Let u, v be a pair of antipodal vertices of G and let P be a path of length d between u and v, where $d = diam\ G$. If $u'$ and $v'$ are the vertices adjacent to u and v respectively in P, then $V - \{u', v'\}$ is a double geodetic dominating set of G. Therefore, $\gamma_{DG}(G) \le p - 2$.

## VI. ON DOUBLE GEODETIC DOMINATION NUMBER OF EDGE ADDED GRAPHS

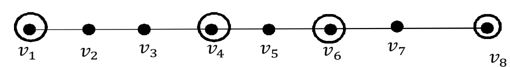
Here, it is studied that how the double geodetic domination number of a non-complete connected graph is affected by addition of a single vertex with one edge incident with this vertex. Throughout this section, $G \circ K_2$ represents the graph obtained from G by adjoining an edge with some vertex of G.

**6.1 Theorem:** Let G be any non-complete connected graphs. Let $G' = G \circ K_2$ be a graph obtained from G by adjoining an edge with some vertex of G. Then, $\gamma_{DG}(G') \le \gamma_{DG}(G) + 1$.

**Proof:** Let $G' = G \circ K_2$. Let $V(G') = V(G) \cup \{u\}$ and $E(G') = E(G) \cup \{uv\}$ for some $v \in V(G)$. If S is any minimum double geodetic dominating set of G, then $S \cup \{u\}$ is a double geodetic dominating set of $G'$. Therefore, $\gamma_{DG}(G') \le |S \cup \{u\}| = |S| + 1 = \gamma_{DG}(G) + 1$.

**6.2 Remark:** In the above theorem, both the upper and lower bounds for $\gamma_{DG}(G')$ are sharp.

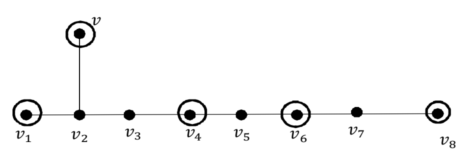
For example, consider $G = P_8, G, G'$ and $G''$ are as in figures 6.2(a), 6.2(b) and 6.2(c) respectively.



**G** Figure 6.2 (a)

$\{v_1, v_4, v_6, v_8\}$ is a double geodetic dominating set of $P_8$.

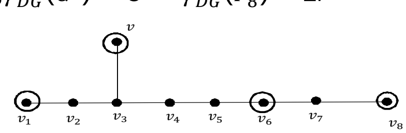
Therefore, $\gamma_{DG}(P_8) = \left\lceil \frac{8-4}{3} \right\rceil + 2 = 2 + 2 = 4$



$G'$ Figure 6.2 (b)

$\{v, v_1, v_4, v_6, v_8\}$ is a double geodetic dominating set of $G'$.

Therefore, $\gamma_{DG}(G') = 5 = \gamma_{DG}(P_8) + 1$.



$G''$ Figure 6.2 (c)

$\{v, v_1, v_5, v_8\}$ is a double geodetic dominating set of $G''$.

Therefore, $\gamma_{DG}(G'') = 4 = \gamma_{DG}(P_8)$.

**6.3 Theorem:** If a vertex is joined by an edge to any vertex of $P_n$, where $n = 3k + 1$ *and* $k \geq 1$, then for the resulting graph $G' = P_n \circ K_2$, $\gamma_{DG}(G') = \gamma_{DG}(P_n) + 1$.

**Proof:**

**Case 1:** suppose $G'$ is the graph obtained form $P_n$ by adding an edge to one of the end vertices of $P_n$.

In this case, $G' \cong P_{n+1}$. Therefore,

$\gamma_{DG}(G') = \gamma_{DG}(P_{n+1}) = \left\lceil\frac{(n+1)-4}{3}\right\rceil + 2 = \left\lceil\frac{3k+2-4}{3}\right\rceil + 2 = \left\lceil\frac{3k-2}{3}\right\rceil + 2 = k + 2$ and

$$\gamma_{DG}(G) = \gamma_{DG}(P_n) = \left\lceil\frac{n-4}{3}\right\rceil + 2 = \left\lceil\frac{3k-3}{3}\right\rceil + 2 = k - 1 + 2 = k + 1.$$

Hence, $\gamma_{DG}(G') = \gamma_{DG}(G) + 1$.

**Case 2:** Suppose $G'$ is obtained by adding an edge to one of the internal of $P_n$.

In this case, the number of end vertices of $G'$ is 3. Therefore, every minimum double geodetic dominating set of $G'$ contains these three end vertices. Clearly, any minimum dominating set of a path of $n-4$ or $n-5$ vertices along with these three end vertices forms a minimum double geodetic dominating set of $G'$ and so $\gamma_{DG}(G') = 3 + \left\lceil\frac{n-4}{3}\right\rceil = 3 + \left\lceil\frac{3k+1-4}{3}\right\rceil = 3 + k - 1 = k + 2 = \gamma_{DG}(G) + 1$, as $\left\lceil\frac{n-4}{3}\right\rceil = \left\lceil\frac{n-5}{3}\right\rceil$ when $n = 3k + 1$.

**6.4 Theorem:** Let G $= P_n$, $n > 3$, and let $n = 3k$. If $G'$ is obtained by adding an edge to one of the end vertices of G, then $\gamma_{DG}(G') = \gamma_{DG}(G)$ and if an edge is added to one of its internal vertices, then $\gamma_{DG}(G') = \gamma_{DG}(G) + 1$.

**Proof:**

**Case 1:** Suppose $G'$ is obtained by adding an edge to one of the end vertices of $G = P_n$.

In this case, $G' \cong P_{n+1}$. Therefore,

$\gamma_{DG}(G') = \gamma_{DG}(P_{n+1}) = \left\lceil\frac{(n+1)-4}{3}\right\rceil + 2 = \left\lceil\frac{3k+1-4}{3}\right\rceil + 2 = k - 1 + 2 = k + 1$ and

$$\gamma_{DG}(G) = \gamma_{DG}(P_n) = \left\lceil\frac{n-4}{3}\right\rceil + 2 = \left\lceil\frac{3k-4}{3}\right\rceil + 2 = \left\lceil\frac{3(k-1)-1}{3}\right\rceil + 2 = k - 1 + 2 = k + 1.$$

Hence, $\gamma_{DG}(G') = \gamma_{DG}(G)$.

**Case 2:** Suppose $G'$ is obtained by adding an edge to any one of the internal vertices of $G = P_n$.

As in theorem 6.4.Clearly, any minimum dominating set of a path of $n-4$ or $n-5$ vertices along with these three end vertices forms a minimum double geodetic dominating set of $G'$ Therefore, $\gamma_{DG}(G') = 3 + \left\lceil\frac{n-4}{3}\right\rceil = 3 + \left\lceil\frac{3k-4}{3}\right\rceil = 3 + \left\lceil\frac{3(k-1)-4}{3}\right\rceil = 3 + k - 1 = k + 2 = \gamma_{DG}(G) + 1$, as $\left\lceil\frac{n-4}{3}\right\rceil = \left\lceil\frac{n-5}{3}\right\rceil$ when $n = 3k$.

## VII. CONCLUSION

Domination not only in Graph Theory but also in real life Problems plays a vital role. It helps to solve many real life situations.

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