Dislocated Quasi B-Metric Space and New Common Fixed Point Results

Dr. S.K. Tiwari[1], Payal vishwakarma[2]
[1, 2] Dept of mathematics
[1, 2] Dr.C.V.Raman University, Kota Bilaspur (C.G)

Abstract*: In this paper, is to study of some fixed point result. Dislocated quasi b-Metric space for different types of contractive combinations. our result extend, generalized , modified, unify some existing result in the literature*.

Keywords: complete dislocated quasi b-metric space, Cauchy sequence ,self mapping and fixed point.

I. INTRODUCTION

Fixed point theory is one of the most dynamic research subjects in non-linear analysis .In this area , the first important and significant result was proved by Banach in 1992 for a contraction mapping in a complete metric space .the well known Banach contraction theorem may be stated as follow .

"Every contraction mapping of a complete metric space X into itself has a unique fixed point ". (Bonsall1962).

In 1906, Frechet introduced the notion of metric space, which is one of the notions of cornerstones of not only mathematics but also several quantitative sciences. Due to its importance and application potential, this notion has been extended,improved and generalized in many different ways. An incomplete list of the results of such an attempt is the following; Quasi symmetric space [6] , A-metric space [2] . S-metric space [10] and so on.

In the field of metric fixed point theory the first fixed point theory the first significant result was proved by Banach in complete metric space which may stated as following:-

"Every contraction mapping of a complete metric space X into itself has a unique fixed point.

Most of the work done in the field of metric fixed point theory after Banach contraction principle involve the continuity of self mapping for different type of contractions.therefore generally a natural question arises whether the conditions of continuity of mapping is essential for the existence of fixed point .this question has been affirmatively answered by Kannan established the following result in which the continuity of mapping is not necessary at each point.

If a mapping $\theta : B_1 \to B_1$ where (B_1, ζ) is a complete metric space and the following conditions holds :-

Then θ has a unique fixed point. The mapping satisfying the above axiom is known is Kannan type of mapping .Kannan [7] provide a new direction for the researchers to work in the area of metric fixed point theory .Almost similar type of contraction condition has been studied by chatterjea [9] whose result may be stated as following :-

Suppose (A_1, ζ) is a complete metric space .A mapping $T_1 : A_1 \to A_1$ Satisfying:
$\zeta(T_1 a_1, T_1 a_2) \leq \delta\, [\zeta(a_1, T_1 a_2) + \varsigma(a_2, T_1 a_1)]$
For all $a_1, a_2 \in A_1$.

Then T_1 has a unique fixed point .the mapping satisfying the above condition is known as chatterjea type of mapping Dass and Gupta [1] generalized Banach contraction conditions in metric space.

The notion of b-metric space was introduced by Czerwik [8] in connection with some problems . concerning with the convergence of non measurable function with respect to measure. Fixed point theorems. Regarding b-metric

Spaces was obtained in [4] and [3] . In 2013, Shukla [11] and generalized the notion of b-metric space and introduced.

The concept of partial b-metric space . Recently Rahman and Sarwar [4] .Further generalized the concept of b-metric space and initiated the notion of dislocated quasi b-metric space.

In the present work ,we have proved some fixed point theorems for generalized type contraction conditions in the setting of dislocated quasi b-metric spaces.which improve ,extend and generalize similar type of fixed point results in dislocated quasi b –metric spaces.

Preliminaries:-

We need the following definitions which may be found in [4] .

1.1Dislocated quasi b-metric space

Definition (1.1.1):-let X be a non empty set and K≥1 be a real number then a mapping d:X×X→[0 ,∞) is called dislocated quasi b-metric space. If for all x ,y , z ∈ X.

(1) d(x , y) = d(y , x) = 0 implies that x = y.
(2) d(x , y)≤ K[d(x ,z) + d(z,y)].

The pair (X ,d) is called dislocated quasi b-metric space .

Remark:-In the definition dislocated quasi b-metric space if k=1 then it becomes (usual) dislocated quasi metric space. therefore every dislocated quasi metric space is dislocated quasi b metric space and every b-metric space is dislocated quasi b –metric space with some coefficient k and zero self distance .However ,the converse is not true as clear from the following example .

Example(1.1.1):- Let X=R and suppose d(x , y) = $|2x - y|^2 + |2x + y|^2$

Then (X , d) is a dislocated quasi b-metric space with the coefficient k=2 . But it is not dislocated quasi b –metric space nor b-metric space.

Remark:- Like dislocated quasi metric space In dislocated quasi b-metric spaces. The distance between similar points need not be zero necessarily as clear from the above example.

1.2 dislocated quasi b-limit

Definition(1.2.1):- A sequence $\{x_n\}$is called dislocated quasi b convergent in (X , d) if for n ≥ N .We have d(x_n ,x) < ∈, where∈ > 0 then x is called the dislocated quasi b limit of the sequence $\{x_n\}$.

1.3 Cauchy sequence

Definition(1.3.1):- A sequence $\{x_n\}$in dislocated quasi b metric space (x , d) is called Cauchy sequence if for ∈>0 there exist n_0 ∈ N, such that for m,n ≥ $n_0$we have d(x_m, x_n)<∈

1.4 Complete

Definition (1.4.1):- A Dislocated quasi b-metric space (x , d) is said to be complete if every Cauchy sequence in X converges to a point of X.

1.5 Continuous

Definition (1.5.1):- Let (X ,d_1) and (Y ,d_2) be a two dislocated quasi b-metric space . A mapping T : X→Y is said to be continuous if for each $\{x_n\}$ which is dislocated quasi b convergent to x_0 in X , the sequence $\{Tx_n\}$ is dislocated quasi b convergent to Tx_0 in Y .

1.6 comparison function:- let (X , d) be a complete metric space and every ∅ contraction T : X→X is a picard's operator.

A map φ : $R_+ \to R_+$ is called comparison function if it satisfies :-

(1) φ is a monotonic increasing function ;
(2) The sequence $\{\varphi^n(t)\}_{n=0}^{\infty}$ converge to 0 for all t∈ R_+ Where φ^n stand for nth iterate of φ.
If φ satisfies :
(3) $\sum_{k=0}^{\infty} \varphi^k(t)$ converge for all t∈ R_+.

Then φ is called (c)- Comparison function.

The following well known results can be seen in [4].

Lemma(1):-Limit of a convergent sequence in dislocated quasi b-metric space is unique.

Lemma(2):-Let (x , d) be a dislocated quasi b-metric space and $\{x_n\}$ be a said to be a sequence in dislocated quasi b-metric space such that

d(x_n ,x_{n+1}) ≤ α d(x_{n-1} ,x_n) …….(1)

for n = 1,2,3,… and $0 \leq \alpha$K < 1 , $\alpha \in [0,1)$; andK is defined in dislocated quasi b – metric space . Then $\{x_n\}$ is a Cauchy sequence in X.

Theorem(1):-Let (x,d) be a complete dislocated quasi b-metric spac. A mapping T : X →Y be a continuous contraction with $\alpha \in [0,1)$

And $0 \leq \alpha$K <1 , where K≥ 1 . Then T has a unique fixed point in X.

Remark:- Like b-metric space dislocated quasi b-metric space is also continuous on its two variables.

Remark:- In dislocated quasi b-metric space the distance between similar points need not to be zero like usual dislocated quasi metric space.

II. MAIN RESULTS

THEOREM(1):- Let (x,d) be a complete dislocated quasi b metric space . Let $T_1 : X \rightarrow X$ and T_2: X→ X be a continuous function on x .for k≥ 1 satisfying

d(T_1x, T_2y) ≤ ∅ d(x,y) …(1)

$\forall x, y \in X$.

Where ∅ is a Comparison function . Then T_1 and T_2 has a unique common fixed point in X.

PROOF:- Let x_0 arbitrary point in X. and $\{x_n\}$ be any sequence in X.

such that $x_0; x_1 = T_1x_0, x_2 = T_1x_1, x_3 = T_1x_2, \ldots, x_{2n+1} = T_1x_{2n}$ …(2)

$x_0; x_1 = T_2x_0, x_2 = T_2x_1, x_3 = T_2x_2, \ldots, x_{2n+2} = T_2x_{2n+1}$…(3)

$\forall n \in N$; now consider to show that$\{x_n\}$is a Cauchy sequence in X.consider

d(x_{2n}, x_{2n+1}) = d(T_1x_{2n-1}, T_2x_{2n}) …(4)

using (1) we have

d$(x_{2n}, x_{2n+1}) \leq$ ∅ d(x_{2n-1}, x_{2n}) …(5)

similarly one can show that

d$(x_{2n-1}, x_{2n}) \leq$ ∅ d(x_{2n-2}, x_{2n-1}) …(6)

putting (4) in (3);

d$(x_{2n}, x_{2n+1}) \leq \emptyset^2d(x_{2n-2}, x_{2n-1})$

proceeding in similar manner we get

d$(x_{2n-1}, x_{2n}) \leq \emptyset^nd(x_0, x_1)$ …(7)

to show that $\{x_{2n}\}$ is a Cauchy sequence .consider m > n and using (d_2)

we have

d$(x_{2n}, x_{2m}) \leq k$d(x_{2n}, x_{2n+1})+d(x_{2n+1}, x_{2n+2}) +…

using (7) the equation become

d$(x_{2n}, x_{2m}) \leq k\emptyset^nd(x_0, x_1)$+$k^2\emptyset^{n+1}d(x_0, x_1) + \cdots$

n, m → ∞

which show that $\{x_n\}$ is a Cauchy sequence in complete dislocated quasi b metric space X . so there exist z ∈ X

now to show that z is the fixed point of T.since x_{2n} →z as n → ∞ using the continuity of T_1 *and* T_2; we have

lim T_1x_{2n}= T_1z and
n→∞

which implies that

lim T_1x_{2n+1}= T_1z
n→∞

similarly we can show that lim T_2x_{2n}= T_2z and
n→ ∞

lim T_2x_{2n+1} = T_2z
n → ∞

taking limit n → ∞; T_1z = z …(8)

and also T_2z = z …(9)

from (8) and (9), we have

T_1z = T_2z = z

so z is the common fixed point of $T_1$and T_2.

UNIQUENESS:- suppose that $T_1$and T_2 has two fixed points z and w . for z≠ w.consider

d(z,w) = d(T_1z, T_2w)

using (1) we have

d(z,w) ≤ ∅ d(z,w)

since ∅ is a comparison function so the above inequality is possible only if d(z,w)= 0 similarly one can show that d(w,z) = 0.so by (d_1)

z = w .hence $T_1$and T_2 has a unique common fixed point in X.

THEOREM (2):- Let (x,d) be a complete dislocated quasi b metric space . Let T : X →X be a continuous self function on x .for k≥ 1 satisfying

d(Tx, Ty) $\leq a_1$d(x,y)+$a_2d(x, Tx) + a_3d(y, Ty) + a_4d(x, Ty) + a_5$d(y,Tx) …(1)

$\forall x, y \in X$,Where$a_1, a_2, a_3, a_4, a_5 \geq 0$ $with ka_1 + ka_2 + ka_3 + ka_4 + a_5 < 1$

Then T has a unique fixed point in X .

PROOF:-

Let x_0 be any arbitrary point in X and $\{x_n\}$ be any sequence in X

such that $x_0; x_1 = Tx_0, x_2 = Tx_1, x_3 = TX_2, \ldots, x_{2n+1} = Tx_{2n}$ ...(2)

now consider to show that $\{x_n\}$ is a Cauchy sequence in X.consider

d(x_{2n}, x_{2n+1}) = d(Tx_{2n-1}, Tx_{2n}) …(3)

using (1) we have

d$(x_{2n}, x_{2n+1}) \leq$
a_1d$(x_{2n-1}, x_{2n}) + a_2d(x_{2n-1}, Tx_{2n-1})$+$a_3d(x_{2n}, Tx_{2n}) + a_4d(x_{2n-1}, Tx_{2n}) + a_5d(x_{2n}, Tx_{2n-1})$

d$(x_{2n}, x_{2n+1}) \leq$
a_1d$(x_{2n-1}, x_{2n}) + a_2d(x_{2n-1}, x_{2n})$+$a_3d(x_{2n}, x_{2n+1}) + a_4d(x_{2n-1}, x_{2n+1}) + a_5d(x_{2n}, x_{2n})$ …(4)

simplifications yields
d(x_{2n}, x_{2n+1}) $\leq \frac{a_1+a_2+a_4}{1-a_3-a_4} d(x_{2n-1}, x_{2n})$
Let h = $\frac{a_1+a_2+a_4}{1-a_3-a_4} < \frac{1}{k}$
So the above inequality become
d(x_{2n}, x_{2n+1}) $\leq h\, d(x_{2n-1}, x_{2n})$
d(x_{2n-1}, x_{2n}) $\leq h\, d(x_{2n-2}, x_{2n-1})$
d(x_{2n}, x_{2n+1}) $\leq h^2 d(x_{2n-2}, x_{2n-1})$ …(5)
similarly proceeding we get ,
d(x_{2n}, x_{2n+1}) $\leq h^n d(x_0, x_1)$ …(6)
now since h < 1/k . taking limit n→ ∞ , $h^n \to 0$
lim (x_{2n}, x_{2m}) =0
n→∞
so by lemma sequence is Cauchy sequence in complete dqb metric space so there must exist u ∈ X *such that*
lim (x_{2n}, u) =0
n→∞
now to show that u is fixed point of T.since $x_{2n} \to u$ *as* $n \to \infty$ using the continuity of T we have
lim Tx_{2n} = Tu
n→∞

which implies that
lim Tx_{2n+1}=Tu
n→∞
thus Tu = u.so u is the fixed point of T.

UNIQUENESS:-Let T have two fixed points with u≠ v then we have
d(u,v) = d(Tu,Tv) $\leq a_1$d(u,v)+$a_2 d(u,Tu) + a_3 d(v,Tv) + a_4 d(u,Tv) + a_5$d(v,Tu)
$\leq a_1$d(u,v)+$a_2 d(u,u) + a_3 d(v,v) + a_4 d(u,v) + a_5$d(v,u)
putting u = v in eq (1) one can easily show that d(u,v) = 0 and d(v,u) = 0 . so by (d_1) we get that u = v .thus fixed point of T is unique.

THEOREM (3):- Let (x,d) be a complete dislocated quasi b metric space .Let T_1 : X→X and T_2 :X→ X be a continuous self function on x .for k≥ 1 satisfying
d($T_1 x, T_2$y) $\leq a_1$d(x,y) $+a_2[d(x,T_1 x) + d(y,T_2 y)] + a_3[d(x,T_1 y)$ +d(y,T_2x)] …(1)
$\forall x, y \in X$, Where $a_1, a_2, a_3 \geq 0$ *with* $ka_1 + (1+k)a_2 + 2(k^2+k)a_3 < 1$
Then T_1 *and* T_2 has a unique common fixed point in X.

PROOF:-
Let x_0 be any arbitrary point in X and $\{x_n\}$ be any sequence in X.
such that $x_0; x_1 = Tx_0, x_2 = Tx_1, x_3 = TX_2, \dots, x_{2n+1} = Tx_{2n}$
$x_0; x_1 = Tx_0, x_2 = Tx_1, x_3 = TX_2, \dots, x_{2n+2} = Tx_{2n+1}$…(2)
now consider to show that $\{x_{2n}\}$ is a Cauchy sequence in X . consider
d(x_{2n}, x_{2n+1}) = d($T_1 x_{2n-1}, T_2 x_{2n}$) …(3)
using (1) we have
d(x_{2n}, x_{2n+1}) ≤
a_1d(x_{2n-1}, x_{2n}) + $a_2[d(x_{2n-1}, T_1 x_{2n-1}) + d(x_{2n}, T_2 x_{2n})]$ + $a_3[d(x_{2n-1}, T_1 x_{2n}) + d(x_{2n}, T_2 x_{2n-1})] \leq$
a_1d(x_{2n-1}, x_{2n}) + $a_2[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})]$ + $a_3[d(x_{2n-1}, x_{2n+1}) + d(x_{2n}, x_{2n})]$ …(4)
using triangular inequality in third term we get,
[1-($a_2 + 2ka_3$)] $d(x_{2n}, x_{2n+1}) \leq (a_1 + a_2 + 2ka_3)$ d(x_{2n-1}, x_{2n})
d(x_{2n}, x_{2n+1}) $\leq \frac{2a_1+a_2}{1-(a_1+2a_2)} d(x_{2n-1}, x_{2n})$
by given restrictions on the constants we have
Let h = $\frac{a_1+a_2+2ka_3}{1-(a_2+2ka_3)} < 1$/h
So the above inequality become
d(x_{2n}, x_{2n+1}) $\leq h\, d(x_{2n-1}, x_{2n})$ ... (5)
so by lemma sequence $\{x_n\}$ is Cauchy sequence in complete dqb metric space so there must exist u ∈ X *such that*
lim (x_{2n}, u) =0
n→∞
now to show that u is fixed point of T . since $x_{2n} \to u$ *as* $n \to \infty$ using the continuity of T_1& T_2 we have
lim $T_1 x_{2n} = T_1$u
n→∞
which implies that
lim $T_1 x_{2n+1} = T_1$u
n→∞
and also
lim $T_2 x_{2n} = T_2$u
n→∞
which implies that
lim $T_2 x_{2n+1} = T_2$u
n→ ∞
T_1u = T_2u = u .so u is the common fixed point of T_1&T_2.

UNIQUENESS:- Let T have two fixed points with u≠ v then we have

d(u,v) = d(T_1u,T_2v) $\leq a_1$d(u,v)+$a_2[d(u,T_1 u) + d(v,T_2 v)] + a_3$[d(u,$T_1$v)+d(v,$T_2 u$]

since u and v are fixed point of T and using given condition in the theorem one can easily get that d(u,u) = 0 and d(v,v) = 0 so finally we get ,

d(u,v)≤ $(a_1 + a_3)$ d(u,v)+a_3d(v,u) …(6)
similarly we can show that
d(v,u)≤ a_3 d(u,v)+$(a_1 + a_3)$ d(v,u) …(7)
adding (7) and (8) we get,
[d(u,v)+d(v,u)] ≤ $(a_1 + 2a_3)[d(u,v) + d(v,u)]$

The above inequality is possible only if d(u,v)+d(v,u) = 0 . which is again possible if d(u,v) = d(v,u) = 0 . so by (d_2) we get that u = v .thus fixed point of T_1& T_2 is unique.

REFERENCES

[1] B.K. Dass and S. Gupta(1975) , an Extension of Banach's Contraction Principle Through Rational Expression , Indian journal of Pure and Applied Mathematics , (6),1455-1458.

[2] M. Abbas (2015) et al. Generalized coupled Common Fixed Point Results in Partially Ordered A-Metric Spaces ,Fixed Point Theory and Applications,64.

[3] M Kir and H. Kizitune(2013),On Some Well Known Fixed Point Theorems in b-Metric Spaces,Turkish Journal of Analysis and Number Theory 1,13-16.

[4] M.U. Rahman and M. Sarwar (2016) , Dislocated Quasi b Metric Space and Fixed Poin Theorems , Electronics Journal of Mathematical Analysis and Applications,4.

[5] M. Sarwar and M.U. Rahman(2015) ,Fixed Point Theorems for Ciric's and Generalized Contractions in b Metric Spaces , International Journal of Analysis and Applications ,7,70-78.

[6] P. S. Kumari and M. Sarwar (2016), Some fixed Point Theorems in Generating space of b-Quasi- Metric Family , SpringerPlus, 5: 268 .

[7] R.Kannan (1968), Some Results on Fixed Points ,Bull.Cal. Math.Soc. India,60,71-76.

[8] S.Czerwik(1993), Contraction Mappings in b Metric Spaces , ActaMath.Inform.Univ. Ostraviensis,1.5-11.

[9] S.K. Chatterjea(1972),Fixed Point Theorems, C. R. AcadBulgura Sci., 25, 727-730.

[10] S.Sedghi (2012) et al., A Generalization of Fixed Point Theorems in S-Metric Spaces,Fixed Point Theorem In S-Metric Spaces , Math . Vesn.,64.

[11] S.Shukla(2013) , Partial b Metric Spaces and Fixed Metric Spaces and Fixed Point Theorems , Mediterr. J Math., June (2013).