Laplace Transforms –A Powerful Tool in Mathematics

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Abstract*-Laplace Transform is powerful technique for engineers and scientist. This paper will provide basic idea of existence of Laplace transform. Reader will able to find solution of ordinary differential equation, simultaneous equation, integral equation using Laplace Transform .Also we discuss method to find exponential matrix using Laplace Transform .*

Keywords-Transform, Exponential order, simultaneous Equation, Integral Equation.

I. INTRODUCTION

Laplace Transform is powerful mathematical tool that used to transform function of variables as x, y, t into parameter(s) [1]. Ordinary Differential Equation are simplified and converted into algebraic equations by use of Laplace Transform [2]. It will convert a function in some domain into a function in another domain without changing the value of the function[3]. Topics covered include the existence of Laplace transforms and inverse Laplace transforms together with applications to ordinary and partial differential equations, integral equations, difference equations.

II. EXISTENCE OF LAPLACE TRANSFORM

If function is piecewise continuous and of exponential order then Laplace Transform of function exist. Laplace Transform can only be used to transform variables that cover a range from zero to ∞.

i) Piecewise continuous

f(t) must be piecewise continuous which means that it must be single valued but can have a finite number of finite isolated discontinuities for t > 0.[4]

ii) **Exponential order**

A function f(t) is of exponential order if there exist finite constant M and α and T such that $|f(t)| < Me^{\alpha t}$ $for\ t > T$[4]. Some examples of functions of exponential order $f(t) = t^2$, $f(t) = 20t + 3$, $f(t) = 2^t$ $f(t) = e^{10t}$[7].Some examples of functions not of exponential order $f(t) = e^{2t^2+1}$, $f(t) = e^{t^3}$[2]

Definition of Laplace Transform

Laplace Transformof f(t) is written as $L(f(t))$ and denoted by $F(s)L(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)dt$.

Inverse Laplace Transform

We refer $f(t)$ as inverse Laplace transform of $F(s)$ and write it as $f(t) = L^{-1}(F(s))$.

Laplace Transform is used to find solution of different type of Equations. We discuss here some solutions obtained by using Laplace Transform.

Solution of Linear Differential Equation with constant coefficient using Laplace Transform

Consider the Equation

$$\frac{dy}{dt} + 3y(t) + 2\int_0^t y(t)\,dt = t, y(0) = 0 \quad (1)$$

Using Laplace Transform both side and Laplace of derivatives and integration we get

$$[sY(s) - y(0)] + 3Y(s) + \frac{2}{s}Y(s) = \frac{1}{S^2}$$

Put $y(0) = 0$ to get

$$Y(s) = \frac{1}{s(s+1)(s+2)} \quad (2)$$

Using Partial fraction method , we have

$$Y(s) = \frac{\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s+2} - \frac{1}{s+1} \quad (3)$$

$y(t)$is obtained using Inverse Laplace Transform as

$$y(t) = \frac{1}{2}t + \frac{1}{2}e^{-2t} - e^{-t} \quad (4)$$

Solution of Simultaneous Differential Equation using Laplace Transform

Consider the Equations

$$\frac{dx}{dt} + y(t) = \sin t \quad (5)$$

$$\frac{dy}{dt} + x(t) = \cos t \quad (6)$$

With the condition $x(0) = 0$ and $y(0) = 2$.

Apply Laplace Transform to equation (5) & using L.T of derivatives (5) can be written as

$$[s\,X(s) - x(0)] + Y(s) = \frac{1}{s^2+1}$$

$$sX(S) + Y(s) = \frac{1}{s^2+1} \qquad (7)$$

Apply Laplace Transform to equation(6) also using L.T of derivatives (6) can be written as

$$[s\,Y(s) - y(0)] + X(s) = \frac{S}{s^2+1}$$

$$[s\,Y(s)] + X(s) = \frac{s}{s^2+1} + 2 \qquad (8)$$

Multiply equation (7) by s

$$s^2X(S) + sY(s) = \frac{s}{s^2+1} \qquad (9)$$

Solving equation (8) and equation (9) we have

$X(S) = \frac{s}{s^2-1}$ and $Y(S) = \frac{2s}{s^2-1} + \frac{1}{s^2+1}$

$x(t)$& $y(t)$is obtained using Inverse Laplace Transform as

$$x(t) = -2\sin ht,\;\; y(t) = \sin t + 2\cos ht \quad (10)$$

Solution of Difference Equation using Laplace Transform

Consider the Equation,

$y(t) - 3\,y(t-1) + 2\,y(t-2) = 1$with condition $y(t) = 0\;if\;\; t < 0$

.

On applying Laplace Transform we have,

$$L\big(y(t)\big) - 3L\big(y(t-1)\big) + 2\,L\big(y(t-2)\big) = L(1) \qquad (11)$$

Now consider

$$L\big(y(t-1)\big) = \int_0^{\infty} e^{-st} y(t-1)dt$$

Substitute $(t-1) = u\,,\;\; dt = du$ we have

$$L\big(y(t-1)\big) = \int_0^{\infty} e^{-s(u+1)} y(u)\,du$$

$$L\big(y(t-1)\big) = e^{-s}\left[\int_{-1}^{0} e^{-su} y(u)du + \int_0^{\infty} e^{-su} y(u)du\right]$$

as $y(u) = 0\;if\;\; u < 0$

$$L\big(y(t-1)\big) = e^{-s}\left[\int_0^{\infty} e^{-su} y(u)\,du\right]$$

$$L\big(y(t-1)\big) = e^{-s}\,Y(s) \qquad (12)$$

Similarly we compute

$$L\big(y(t-2)\big) = e^{-2s}\,Y(s) \qquad (13)$$

Using values of equation (12) &(13) in (11) we get

$$Y(s) = \frac{1}{s(1-3e^{-s}+2e^{-2s})} \qquad (14)$$

$$Y(s) = \frac{1}{s}\left[\frac{2}{1-2e^{-s}} - \frac{1}{1-e^{-s}}\right] \qquad (15)$$

$$Y(s) = \frac{1}{s}\left[\sum_0^{\infty} e^{-ns}\, 2^{n+1}\right] - \frac{1}{s}\sum_0^{\infty} e^{-ns} \qquad (16)$$

Taking inverse transform we have

$$y(t) = 2^{[t]+2} - [t] - 3$$

Where $[t]$ is greatest integer less than or equal to t.

To find Exponential (e^{At}) of matrix A using Laplace Transform

If $A = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$, Applying the property $L(e^{At}) = \frac{1}{s-A}$, in matrix sense$L(e^{At}) = \frac{1}{sI-A}$, or $L(e^{At}) = (sI-A)^{-1}$, can be rewritten as

$$(e^{At}) = L^{-1}((sI-A)^{-1}) \qquad (17)$$

Determinant of $[sI - A]$will be given by

$$|sI - A| = s^2 - 4s + 3 \qquad (18)$$

Adjoint of $[sI - A] = \begin{bmatrix} s-3 & -1 \\ 0 & s-1 \end{bmatrix} \qquad (19)$

$$[sI - A]^{-1} = \begin{bmatrix} \frac{s-3}{s^2-4s+3} & \frac{-1}{s^2-4s+3} \\ 0 & \frac{s-1}{s^2-4s+3} \end{bmatrix} \qquad (20)$$

Consider

$$L^{-1}\left[\frac{s-3}{s^2-4s+3}\right] = L^{-1}\left[\frac{s-2-1}{(s-2)^2-1}\right] = e^{2t}(\cos ht - \sin ht) \quad (21)$$

$$L^{-1}\left[\frac{-1}{s^2-4s+3}\right] = L^{-1}\left[\frac{-1}{(s-2)^2-1}\right] = -e^{2t}(\sin ht) \quad (22)$$

$$L^{-1}\left[\frac{s-1}{s^2-4s+3}\right] = L^{-1}\left[\frac{s-2+1}{(s-2)^2-1}\right] = e^{2t}(\cos ht + \sin ht) \quad (23)$$

Using equation (17) & (21),(22),(23) we obtain

$$(e^{At}) = L^{-1}((sI-A)^{-1})$$

$$(e^{At}) = \begin{bmatrix} e^{2t}(\cos ht - \sin ht) & -e^{2t}(\sin ht) \\ 0 & e^{2t}(\cos ht + \sin ht) \end{bmatrix}$$

Solution of Integral equation of convolution type using Laplace Transform

Consider the equation

$$y(t) = t + 2\int_0^t \cos(t-u)\, y(u)du \qquad (24)$$

$$y(t) = t + [y(t) * \cos t]$$

Applying Laplace Transform

$$Y(s) = \left[\frac{s^2+1}{s^2(s-1)^2}\right] \qquad (25)$$

Applying Inverse Laplace Transform we get $y(t)$

$$y(t) = t + 2 + 2e^{t}(t-1) \tag{26}$$

III. CONCLUSION

Laplace Transform method provides easy mode for solution of many problems in various fields in Engineering and sciences. Topics covered includes method to find solution of Linear Differential Equation with constant coefficient, Simultaneous Differential Equation, Difference Equation, Integral equation of convolution type. In this addresses issues like exponential of matrix are discussed which is rarely considered. Further we can extend the text to find solution of partial differential equation using Laplace Transform. We summarized and conclude that Laplace Transform is a powerful tool in Mathematics.

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