On Periodicity and Stability of Solutions of First Order Neutral Differential Equation

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Abstract*- In this paper, we discuss periodicity and stability of solutions of first order neutral differential equation involving piecewise constant deviating arguments. Examples are given in support of the results.*

Keywords*-* Neutral differential equation, periodicity, stability.

I. INTRODUCTION

There has been considerable interest in the study of periodicity and stability of neutral differential equations. This is due to the importance of neutral differential equations in representing biological systems, processes including steam or water pipes heat exchangers, distributed networks containing lossless transmission lines, etc. [9] These time delays may cause instability and poor performance of the system. So the delay differential equations gives a better description of the system than the ordinary differential equations. In [1], [3], [4], [7] delay differential equations with piecewise constant arguments are studied. In [6], [11], [13],[14], [15] existence of positive periodic solutions of first order neutral differential equation is studied, while in [5], [8], [10],[12], [14] stability analysis of neutral differential equations are studied. The purpose of this paper is to study periodicity and stability of solution of neutral differential equation (NDE) of the type:

$$\frac{d}{dt}[x(t) - a(t)\,x([t])] = p(t)f\big(x([t])\big), \quad t \geq 0. \tag{1}$$

where $a \in C'\big([0,\infty),(0,\infty)\big), p \in C\big(R,(0,\infty)\big), f$ is a continuous function from R^2 to R and [.] denotes greatest integer function.

The paper is organized as follows: In section 2, we give some prerequisites required for proving the main theorem. In section 3 and 4, we give the main results on the periodicity and stability of the NDE (1) respectively. In section 5, we give examples in support of the results.

II. PREREQUISITES

In this section we present the preliminaries required for the main results of this paper. First, we define a solution of (1).

Definition 2.1*: A solution of equation (1) on* $[0,\infty)$ *is a function* (t) *that satisfies the initial condition* $x(0) = x_0$ *and the conditions:*

- $x(t)$ *is continuous on* $[0,\infty)$.
- *The derivative* $x'(t)$ *exist at each point* $t \in [0,\infty)$, *with the possible exception of the points* $[t] \in [0,\infty)$ *where one sided derivative exists.*
- *Equation (1) is satisfied on each interval* $[n, n+1) \subset [0,\infty)$, *with integral end points.*

We require following results.

Lemma2.2 *(Krasnoselskii's fixed point theorem)* [9]:

Let K *be a nonempty complete convex subset of a normed linear space X. Let* T *be a continuous mapping of* K *into a compact subset of* X. *Let* $S: K \to X$ *be a contraction mapping with Lipschitz constant* α *and let* $Tx + Sy \in K$ *for all* $x, y \in K$. *Then there is a point* $u \in K$ *such that* $Tu + Su = u$.

Lemma 2.3*(Contraction Mapping Theorem)* [8]:

Let F *be a continuous mapping of a complete metric space X into itself, such that* F^k *is a contraction mapping of* X *for some positive integer k. Then* F *has a unique fixed point.*

Theorem 2.4 The unique solution of (1) with initial condition

$$x(0) = x_0, \tag{2}$$

on $[0, T]$ is given by

$$x(t) = \sum_{j=0}^{[t]} [\Pi_{i=j}^{[t]-1} A(i, i+1)] F(j-1, j). \tag{3}$$

where $F(-1,0) = x(0)$ and for $j \geq 1$.

$$F(j-1,j)=\int_{j-1}^{j}\{f(u,x(j-1))-p(u)a(u)x(j-1)\}e^{-\int_u^t p(s)\,ds}\,du,$$

$$A(i,i+1)=[a(i+1)+(1-a(i))e^{-\int_0^t p(s)\,ds}].$$

Proof: The proof can be easily obtained by the method of steps.

Let Γ be the set of all continuous periodic scalar functions $x(t)$ with period $\omega(>0)$. Then, $(\Gamma, ||\cdot||)$ is a Banach space with supremum norm

$$||x|| = Sup_{t\in R}|x(t)| = Sup_{t\in[0,T]}|x(t)|. \qquad (4)$$

Let $\kappa, \sigma, \mu, \gamma$ be positive constants. We assume the following conditions:

[V1] $a(t+\omega)=a(t)$ and $p(t+\omega)=p(t)$.

[V2] $f(t,x([t]))$ is a periodic function in t with period ω.

[V3] $max_{t\in[0,\omega]}|e^{\int_0^\omega p(s)\,ds}-1|=\frac{1}{\mu}\neq 0.$

[V4] The following condition is taken from [2]. η be a step function defined from R to R satisfying (ω, l) property. i.e. $\eta_i+\omega=\eta_{i+l}$ where $i=\{0,1,2,3,\dots\}$ and $l\in N$.

[V5] $||a(t)||\le\kappa,\ ||p(t)||\le\sigma\ ,\ ||e^{-\int_u^t p(s)\,ds}||\le\gamma.$

[V6] function f satisfies the following globally Lipschitz condition:

$$||f(t,x)-f(t,y)||\le L_1||x-y||.$$

For the proof of the main theorem on periodicity, we require the following lemma.

Lemma 2.5: *Let conditions [V1]-[V4] holds. If $x\in\Gamma$, then x is a solution of (1) iff*

$$x(t)=a(t)x([t])+\mu\int_t^{t+\omega}f(u,x([u]))e^{-\int_u^t p(s)\,ds}\,du - \mu\int_t^{t+\omega}p(u)a(u)x([u])e^{-\int_u^t p(s)\,ds}\,du. \qquad (5)$$

Proof: Let $x(t)\in\Gamma$ be a solution of (1). Then

$$[x(t)-a(t)x([t])]'+p(t)[x(t)-a(t)x([t])] = f(t,x([t]))-p(t)a(t)x([t]).$$

Multiplying both sides with $e^{\int_0^t p(s)\,ds}$ we get,

$$\{[x(t)-a(t)x([t])]e^{\int_0^t p(s)\,ds}\}' = \{f(t,x([t]))-p(t)a(t)x([t])\}e^{\int_0^t p(s)\,ds}.$$

Integrating from t to $t+\omega$,

$$\begin{aligned} &x(t+\omega)-a(t+\omega)x([t+\omega])e^{\int_0^{t+\omega} p(s)\,ds} \\ &- x(t)+a(t)x([t])e^{\int_0^t p(s)\,ds} \\ &= \int_t^{t+\omega}[f(u,x([u]))-p(u)a(u)x([u])]e^{\int_0^u p(s)\,ds}\,du. \end{aligned}$$

From [V1] , [V4] and dividing by $e^{\int_0^t p(s)\,ds}$ we have,

$$\begin{aligned} &[x(t)-a(t)x([t])](e^{\int_t^{t+\omega} p(s)\,ds}-1) \\ &= \int_t^{t+\omega}[f(u,x([u]))]e^{-\int_u^t p(s)\,ds}\,du \\ &\quad - \int_t^{t+\omega}[p(u)a(u)x([u])]e^{-\int_u^t p(s)\,ds}\,du. \end{aligned}$$

From [V1] and [V3],

$$\begin{aligned} &[x(t)-a(t)x([t])]\frac{1}{\mu} \\ &= \int_t^{t+\omega}[f(u,x([u]))]e^{-\int_u^t p(s)\,ds}\,du \\ &\quad - \int_t^{t+\omega}[p(u)a(u)x([u])]e^{-\int_u^t p(s)\,ds}\,du. \end{aligned}$$

$$\begin{aligned} x(t) &= a(t)x([t]) \\ &+\mu\int_t^{t+\omega}f(u,x([u]))e^{-\int_u^t p(s)\,ds}\,du \\ &-\mu\int_t^{t+\omega}p(u)a(u)x([u])e^{-\int_u^t p(s)\,ds}\,du. \end{aligned}$$

Hence (5) holds.

Now if (5) holds, then $x(t)$ is a solution of equation (1). Hence the proof.

We now define a mapping G by

$$\begin{aligned} (Gy)(t) &= a(t)y([t])+\mu\int_t^{t+\omega}[f(u,y([u])]e^{-\int_u^t p(s)\,ds}\,du \\ &-\mu\int_t^{t+\omega}[p(u)a(u)y([u])]e^{-\int_u^t p(s)\,ds}\,du, \qquad (6) \end{aligned}$$

where $y \in \Gamma$, then $(Gy)(t)$ is periodic.
We write

$$(Gy)(t) = (Dy)(t) + (Cy)(t),$$

where $C, D : \Gamma \to \Gamma$ are given by:
$(Dy)(t) = a(t)y([t])$ and

$$(Cy)(t) = \mu \int_t^{t+\omega} f(u, y([u]))e^{-\int_u^t p(s)\,ds}\,du - \mu \int_t^{t+\omega} p(u)a(u)y([u])e^{-\int_u^t p(s)\,ds}\,du.$$

Remark 2.6:

1. *C is continuous and Cy is contained in a compact set.*
2. *For $\kappa < 1$, D is a contraction.*

III. PERIODICITY

In this section we give results for ω - periodicity of the solution of equation (1)

Theorem 3.1: *Suppose conditions [V1]-[V6] holds. Let M be a positive constant satisfying the inequality $N + \kappa M \leq M$, where $N = \mu\gamma[\sigma\kappa M + L_1 M + \beta]\omega$ and $\kappa < 1$. Consider $K = \{y \in \Gamma : \|y\| \leq M\}$, then (1) has a solution in K.*

Proof: Consider $D, C : K \to \Gamma$, then by Remark 2.6 we have,

- C is continuous and Cy is contained in a compact set.
- D is a contraction.

Observe that,

$$\begin{aligned} \|f(u, y([u]))\| &= \|f(u, y([u])) - f(u, 0) + f(u, 0)\|, \\ &\leq \|f(u, y([u])) - f(u, 0)\| + \|f(u, 0)\|, \\ &\leq L_1(y([u])) + \beta, \end{aligned}$$

where $\beta = Sup_{t\epsilon[0,\omega)} f(t, y(t))$.

$$\begin{aligned} \|(Cy)(t)\| &= \|\mu \int_t^{t+\omega} [f(t, y([u]))]e^{-\int_u^t p(s)\,ds}\,du - \mu \int_t^{t+\omega} [p(u)a(u)y([u])]e^{-\int_u^t p(s)\,ds}\,du\|, \\ &\leq \mu\gamma[\sigma\kappa M + L_1 M + \beta]\omega = N. \end{aligned}$$

Therefore,

$$\begin{aligned} \|Cy + Dz\| &\leq \|Cy\| + \|Dz\|, \\ &\leq N + \kappa M, \\ &\leq M. \end{aligned}$$

Hence, using Krasnoselskii's theorem there exists a fixed point $x \epsilon K$ such that $x = Cx + Dx$ and this fixed point is ω -periodic solution of (1) by Lemma 2.5.

Corollary 3.2 *Suppose conditions [V1]-[V6] holds and $\kappa < 1$. If $\kappa + \mu\gamma[\sigma\kappa + L_1]\omega < 1$, then the equation (1) has a unique ω-periodic solution.*

Proof: Let us consider $y, z \in \Gamma$ be two the solution of (1). Let the function G be defined by (6). Then,

$$\begin{aligned} \|Gy - Gz\| \leq\ & \|a(t)y([t]) \\ & + \mu \int_t^{t+\omega} [f(u, y([u]))]e^{-\int_u^t p(s)\,ds}\,du \\ & - \mu \int_t^{t+\omega} [p(u)a(u)y([u])]e^{-\int_u^t p(s)\,ds}\,du \\ & - \{a(t)z([t]) + \mu \int_t^{t+\omega} [f(u, z([u]))]e^{-\int_u^t p(s)\,ds}\,du\} \\ & - \mu \int_t^{t+\omega} [p(u)a(u)z([u])]e^{-\int_u^t p(s)\,ds}\,du\}\| \\ \leq\ & (\kappa + \alpha\gamma[\sigma\kappa + L_1]\omega)\|y([t]) - z([t])\|. \end{aligned}$$

Hence, by contraction mapping theorem we get $y = z$, which completes the proof.

IV. STABILITY

We now establish a result giving sufficient condition for stability using fixed point theorem. Consider the equation (1) with the initial function $x(t) = y(t)$ for $t\epsilon(-\infty, 0]$. We will use the contraction mapping principle to prove asymptotic stability of neutral differential equation with piecewise constant argument. We need the following definition.

Definition 4.1 Let $y(t) : [-\infty, 0] \to R$ *be a given continuous bounded initial function. We say $x(t) = x(t, 0, y)$ is a solution of (1) if $x(t) = y(t)$ for $t\epsilon(-\infty, 0]$ and satisfies (1) for $t \geq 0$. We say that the zero solution of (1) is stable at t_0 if for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that $y : (-\infty, 0] \to R$ with $\|y\| < \delta$ on $(-\infty, t_0]$ implies $|x(t, t_0, y)| < \varepsilon$.*

Without lost of generality we will state and prove the result for $t_0 = 0$.

Let $C([n, n+1), R)$ be set of continuous bounded functions with the norm defined by $\|y_n\| = Sup_{t\in[n,n+1)} |y_n(t)|$. Then $C([n, n+1), R)$ is a Banach Space.

We now define the subset of $C([n, n+1), R)$ as follows:
$K_n = \{y_n = y_n(t) \in C([n, n+1), R): y_n(t) \to 0 \; as \; t \to \infty\}$,
the($K_n, ||\cdot||$) is a Banach space.

Let

[V7] $\int_{[t]}^{t} p(s)\,ds > 0, lim_{t\to\infty} e^{-\int_{[t]}^{t} p(s)\,ds} = 0.$

[V8] $lim_{t\to\infty} a(t) = 0.$

[V9] $[\kappa + \sigma\gamma(L_1 + 1 + \kappa)] < 1.$

Theorem 4.2: *If [V4]-[V9] holds, then every solution $x(t, 0, y)$ of (1) with small continuous initial function $y(t)$ is bounded and approaches zero as $t \to \infty$. Moreover the zero solution is stable at $t_0 = 0$.*

Proof: Consider map $P_n: K_n \to C([n, n+1), R)$ defined as follows:

$If \; t = n \; then \; (P_n)(y_n(t)) = (P_{n-1})(y_n(n)).$

for $t \in [n, n+1)$

$$(P_n)(y_n(t)) = a(t)y_n(n) + [y_n(n) - a(n)y_n(n)]e^{-\int_n^t p(s)\,ds} + \int_n^t [f(u, y_n(n) - p(u)a(u)y_n(n)]e^{-\int_u^t p(s)\,ds}\,du.$$

For $y_n \in K_n$ with $||y_n|| \le \delta$ for $\delta > 0$, $P_n(y_n)$ is continuous. Also,

$$\begin{aligned} ||P_n(y_n(t)|| &= ||a(t)y_n(n) + [y_n(n) - a(n)y_n(n)]e^{-\int_n^t p(s)\,ds} \\ &\quad + \int_n^t [f(u, y_n(n) - p(u)a(u)y_n(n)]e^{-\int_u^t p(s)\,ds}\,du||, \\ &\le [\kappa + (1+\kappa)\gamma + \gamma(\sigma\kappa + L_1)]\delta + \beta\gamma, \\ &\le \varepsilon. \end{aligned}$$

where $\beta = Sup_{t\epsilon[0,\omega)} f(t, y_n(t))$ and $[\kappa + (1+\kappa)\gamma + \gamma(\sigma\kappa + L_1]\delta + \beta\gamma = \varepsilon$ and which gives $P_n(y_n)$ is bounded. Also,

$$||lim_{t\to\infty} P_n y_n(t)|| = 0.$$

Now we will show that $P_n(y_n(t))$ is a contraction under the supremum norm.

Let $\zeta, \eta \in K_n$

$$\begin{aligned} ||P_n(\zeta(t)) - P_n(\eta(t))|| &= [\kappa + (1+\kappa)\gamma \\ &\quad + \gamma(\sigma\kappa + L_1]||\zeta(n) - \eta(n)||, \\ &\le L||\zeta(n) - \eta(n)||. \end{aligned}$$

where $L = [\kappa + (1+\kappa)\gamma + \gamma(\sigma\kappa + L_1] < 1.$ Thus, by the contraction mapping principle P_n has a unique fixed point in K_n which shows that (1) is bounded and tends to zero as $t \to \infty$. Hence the zero solution is stable.

V. EXAMPLES

In this section we provide examples in support of the two results established in section 3 and 4.

Example 1:
Consider the following equation,

$$\frac{d}{dt}[x(t) - \frac{\sin(\omega t)}{100}x([t])] = -\cos(\omega t)x(t) - \frac{\omega\cos\omega t}{100}x([t]),$$

where $t \ge 0$, ω is positive constant.

Here $a(t) = \frac{\sin(\omega t)}{100}$, $p(t) = cos\omega t$ and

$f(t, x([t])) = -\frac{\omega\cos\omega t}{100}x([t]).$

Clearly $||a(t)|| \le \kappa = 0.02$, $||p(t)|| \le \sigma = 1$ and

f is periodic and satisfies globally Lipschitz condition.
The solution $x(t) = e^{\frac{-\sin\omega t}{\omega}}$ which is periodic with period $\frac{2\pi}{\omega}$.

Example2 : Consider the following equation,

$$\frac{d}{dt}[x(t) - \frac{0.85}{t}x([t])] = -\omega x(t) + (\frac{0.85}{t^2})x([t]),$$

Here $t \ge 0$, and $0 < \omega < 1$, $a(t) = \frac{0.85}{t}$, $p(t) = \omega$ and

$f(t, x([t])) = (\frac{0.85}{t^2})x([t]).$

Clearly $||a(t)|| \le \kappa$, $||p(t)|| \le \sigma = \omega.$

Also, $lim_{t\to\infty} a(t) = 0$ and

$lim_{t\to\infty} e^{-\int_{[t]}^{t} p(u)\,du} = 0.$
Hence the solution $x(t) = e^{-\omega t}$ is asymptotically stable.

VI. CONCLUSION

Periodicity of solutions of first order NDE with piecewise constant deviating argument of the type (1) is obtained under suitable conditions by employing Krasnoselskii's fixed point theorem. The stability of a zero solution of (1) is discussed by using the contraction mapping theorem.

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