Circular Chromatic Number

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Abstract- *This paper attempts to study the circular chromatic number $\chi_c(G)$ of a graph G was introduced by Vince (also known as the Star Chromatic number) is a natural generalization of the chromatic number of a graph. In this paperwe concentrating on the relations among the circular chromatic number, the chromatic number and some other parameters.We prove that if an $(m+1)$ critical graph has large girth, then its star chromatic number is close to m.*

Keywords- Circular chromatic number,coloring, m-colorable.

I. INTRODUCTION

Let *C* be a circle of (Euclidean) length *r*. An *r*-circular coloring of a graph *G* is a mapping *c* which assigns to each vertex *x* of *G* an open unit length arc $c(x)$ of *C*, such that for every edge (x,y) of *G*, $c(x)\cap c(y)=\phi$. We say a graph *G* is *r*-circular colorable if there is an *r*-circular coloring of *G*.

The circular chromatic number of a graph denote by $\chi_c(G)$, is defined as, $\chi_c(G)=\inf\{r: G \text{ is } r-\text{circular colorable}\}$.

It is easy to see that if $\chi_c(G)=r$, then for any $r'\geq r$, there is an r'-circular coloring of *G*. nother
trivial observation is that if *H* is a subgroup of *G* then $\chi_c(H)\leq\chi_c(G)$.

Lemma:1

Suppose *G* is a finite graph, and that *c* is an *r*-circular coloring of *G*. If $D_c(G)$ is acyclic then there is an r'-circular coloring c' of *G* such that $r'<r$ and $D_c(G)$ contains a directed cycle.

Proof: Suppose $D_c(G)$ is acyclic.

For each vertex *x*, define the level $l(x)$ to be the length of a longest directed path in *D* which ends at *x*. (since $D_c(G)$ is acyclic, such a path exists).Let x_0 be a vertex with maximum level. Then the interval $c(x_0)$ can be shifted to the right (i.e. to the clockwise direction) by a small distance, without violating the condition that adjacent vertices are assigned to disjoint intervals.

After the shifting, the vertex x_0 because an isolated vertex in the corresponding digraph. By repeating this process we obtain another *r*-circular coloring c'' such that the digraph $D_{c''}(G)$ has no arcs.

Therefore each interval $c''(x)$ can be stretched to a longer interval, say an interval of length $s>1$ and still satisfying the condition that adjacent vertex corresponds to disjoint intervals. We now uniformly shrink the circle *C* into a circle C' of r/s .Each interval of *C* of length *s* is shrink to an interval of C' of length 1. Thus we obtain an r/s -circular coloring of *G*.
We may repeat this process, if needed, to obtain r'-circular coloring c' with $r'<r$ and such that $D_{c'}(G)$ contains a directed cycle.
This completes the proof of lemma.

Lemma:2

If *G* is *r*-circular colorable and for every *r*-circular coloring *c* of *G*, $D_c(G)$ contains a directed cycle, then $\chi_c(G)=r$.
Therefore a graph *G* has $\chi_c(G)=r$ if and only if *G* is *r*-circular colorable and for every *r*-circular coloring *c* of *G*, $D_c(G)$ contains a directed cycle.

Proof: For a $k\geq d$, a (k,d)- coloring of a graph $G=(V,E)$ is a mapping $f:V\to\{0,1,\cdots,k-1\}$ such that for every edge *xy* of *G*.

$d \le |f(x) - f(y)| \le k - d$.

As shown in the next section, the circular chromatic number can be defined as $\chi_c(G) = \min\{k/d : G \text{ admits a } (k,d)\text{- coloring}\}$.For a (k,d)- coloring ϕ of a graph *G*, Let $D_\phi(G)$ be the digraph with vertex set $V(G)$ and *xy* is a directed edge of $D_\phi(G)$ if and only if *xy* is an edge of *G* and $\phi(y) - \phi(x) = d \pmod k$.

The digraph $D_\phi(G)$ is analog to $D_c(G)$ for a k/d - circular coloring and they have similar properties.

In particular, lemma 3 is true for $D_\phi(G)$ i.e. we have the following lemma, which is proved in [2].

Theorem:

Suppose *G* is a graph and $X \subset V(G)$ is a subset of the vertex set of *G*. If *G* is (k,d)- colorable, and for any (k,d)- coloring *f* of *G* $f(X) = \{0,1,\cdots,p-1\}$ and the restriction $f \mid X$ is unique up to a permutation of the colors, then $\chi(G) = k/d$.

Proof:

Assume *G* is a graph satisfying the conditions above. Since *G* is (k,d)- colorable, we have $\chi_c(G) \le k/d$ .Assume to the contrary that $\chi_c(G) < k/d$.

Then ,*G* has a (k,d)- coloring ϕ such that $D_\phi(G)$ is acyclic.

We define the level of a vertex *v* of $D_\phi(G)$ to be the length of a longest directed path ending at *v* (such a path exists, because $D_\phi(G)$ is acyclic).

Let $v^* \in X$ be a vertex of X whose level is maximum. Let ϕ' be the mapping defined as follows: if there is a directed path in $D_\phi(G)$ from v^* to *x*, then $\phi'(x) = \phi(x) + 1 \pmod k$

Otherwise, $\phi'(x) = \phi(x)$.

Then it is straight forward to verify that ϕ' is also a (k,d)- coloring of *G*.Moreover, $\phi' \mid X = \phi \mid X$, except that $\phi'(v^*) \ne \phi(v^*)$. Therefore $\phi' \mid X$ cannot be obtained from $\phi \mid X$ by a permutation of colors. (Because some vertices which are colored by different colors by ϕ are now colored by the same color by ϕ'). This is in contrary to our assumption.

II. EQUIVALENT FORMULATIONS

The circular chromatic number $\chi_c(G)$ of a graph was introduced by Vince in 1988 as “the star-chromatic number”.

However, the definition given in the previous section is not the original definition of Vince, but an equivalent definition given by the author in a slightly different form. The original definition of Vince is as follows:For two integers $1 \le d \le k$, a (k,d)- coloring of a graph *G* is a coloring *c* of the vertices of *G* with colors $\{0,1,2,\cdots,k-1\}$ such that,

$(x,y) \in E(G) \Rightarrow d \le |c(x) - c(y)| \le k - d$

The circular chromatic number is defined as,

$\chi_c(G) = \inf\{k/d : \text{ there is a } (k,d) - \text{ coloring of } G\}$

For any integer *k*, a $(k,1)$- coloring of a graph *G* is just an ordinary *k*- coloring of *G*.

Suppose *c* is $a(k,d)$- coloring of *G*.

Let $c' : V(G) \to [0, k/d)$ be the mapping defined as $c'(x) = c(x) \mid d$ then for every edge (x,y) of *G*, we have $1 \le |c'(x) - c'(y)| \le k/d - 1$.

Therefore $a(k,d)$- coloring of *G* corresponds to a k/d - circular coloring of *G*.

On the other hand, it is straight forward to verify that if c' is a k/d - circular coloring of *G* (viewed as a mapping from $V(G)$ to $[0,r)$ then the mapping *c* is defined as $c(x) = \left[c'(x)d\right]$ is a (k,d) - coloring of *G*.

III. GRAPHS *G* FOR WHICH

$$\chi_c(G)=\chi(G)$$

It was shown by Guichard [2]that is in *NP*-hard to determine whether or not an arbitrary graph *G*satisfies $\chi_c(G)=\chi(G)$.

Indeed, using an oracle which determine whether or not an arbitrary graph *G*satisfies $\chi_c(G)=\chi(G)$. We can easily determine the chromatic number of a graph *G* as follows.

Let $G\bigcup H$ denote the disjoint union of graphs *G* and *H* and let G_k^d denote the graph with vertex set $\{0,1,\cdots,k-1\}$ and in which *i* is adjacent to *j*when $d\le|i-j|\le k-d$.

It was shown in [1] that $\chi_c(G_k^d)=k/d$. Using this fact, it is straight forward to verify that a graph *G* is *n*-colorable if and only if the following two statements are true.

i) $\chi_c(G\bigcup K_n)=\chi(G\bigcup K_n)$

ii) $\chi_c\left(G\bigcup G_{2n+1}^2\right)\ne\chi\left(G\bigcup G_{2n+1}^2\right)$

Theorem:

Suppose $\chi(G)=n$. If there is a non-trivial subset *A* of *V* (i.e., $A\ne V$ and $A\ne\phi$) such that for any *n*-coloring *c* of *G*, each color class *X* of *c* is either contained in *A* or disjoint from *A*, then $\chi_c(G)=\chi(G)$.

Proof:

Assume that *A* is a subset of *V* satisfying the condition above and assume to the contrary of theorem that $\chi_c(G)=r<n$.

Let *c* be an *r*- circular coloring of *G*. First we show that for any $x\in A,\ y\in V-A,\ c(x)\bigcap c(y)=\phi$.

Otherwise, let $p\in c(x)\bigcap c(y)$ for some $x\in A$ and $y\in V-A$.

Starting from the point *p*, we evenly put *n* points $p=p_1,p_2,\cdots,p_n$ on the circle *C*.

Thus the length of the arc from p_i to p_{i+1} is $r/n<1$.

Therefore each arc $c(z)$ contains at least one of the points $p_1,p_2,\cdots,p_n$.

We color a vertex *z* of *G* with color *i* for some $p_i\in c(z)$ and in particular, color *x* and *y* with color 1.

This is an *n* coloring of *G* which has a color class, the class with color 1, that is neither contained in *A* nor disjoint from *A*. Contrary to our assumption.

Let $P=\{p\in c;\,p\in c(x)\text{ for some }x\in A\}$

Then $c(y)\bigcap P=\phi$ for any $y\in V-A$.

As *A* is a non-trivial subset of *V*, we know that *P* is a non trivial subset of *C*.

Let *q* be a boundary point of *P*, then it is easy to see that $q\notin c(z)$ for any $z\in V$ (Note that each arc $c(z)$ is an open subset of *C*).

Therefore we may cut the circle *C* at*q* to obtain an *r*-interval coloring of *G*, contrary to our assumption that $\chi(G)=n>r$.

Hence the proof.

Theorem:

If $\chi_c(G)=r,\ e=xy$ is an edge of *G* and $(Q:a,b)$ is a strong *r*-circular superedge, then by replacing the edge *e*by $(Q:a,b)$, the resulting graph $G(e,(Q:a,b))$ also has circular chromatic number *r*.

Proof:

Let *f* be a (p,q)- coloring of *G*.

Then $|f(x)-f(y)|_p\ge q$.

By the definition of a strong r-circular super edge, the coloring *f* can be extended to a (p,q)- coloring of $G(e,(Q:a,b))$.

Therefore $\chi_c\left(G(e,(Q:a,b))\right)\le p/q$.

It remains to show that $\chi_c\left(G(e,Q;a,b)\right)\ge p/q$.

Assume to the contrary that there is an $\in>0$ and there is an $(r-\in)$ circular coloring *c*of $G(e,(Q:a,b))$.

By the definition of a strong *r*- circular super edge, we know that $c(a)\bigcap c(b)=\phi$.

Hence *c* is an $(r-\varepsilon)$-circular coloring of *G*. Contrary to the assumption that $\chi_c(G)=r$.
Hence the proof.

Theorem:

For any rational number $r=p/q\geq 3$, and for any integer $g\geq 3$, there is a graph *G* of grith at least *g* and $\chi_c(G)=r$.

Proof:

If there is a strong *r*-circular suberedge $(Q:a,b)$ such that *Q* has girth at least *g* and the distance between *a* and *b* is at least *g*, then we replace each edge of G_p^q by a copy of $(Q:a,b)$. Denote the resulting graph by *G*.

Then it follows from theorem , that $\chi_c(G)=p/q$ on the other hand, it is easy to see that *G* has girth at least *g*.
Thus it remains to show there is a strong *r*-circular superedge $(Q:a,b)$ suchthat *Q* has girth at least *g* and the distance between *a* and*b* is at least *g*.
We shall only consider the case that $r=n$ is an integer. The case $r\geq 3$ is non-integer is technicallymore difficult we shall sketch the idea of a construction below.
(For $2<r<3$, no construction of a strong *r*-circular super edge is known, although the existence can be proved by probabilistic method).

Let *H* be a graph of girth at least *g*with $\chi(H)=n+1$. Moreover for any edge $e=aa'$ of *H*, $\chi(H-e)=n$.(There are a few known methods of constructing such graphs) Delete an edge aa' and add a new vertex *b* and connect *b* to a^1 by an edge.Denote the resulting graph by *Q*.

We shall show that $(Q:a,b)$ is the required strong *n*-circular super edge.
Obviously *Q* has girth at least *g* and *a*, *b* has distance at least *g*.
Since *H* is not *n* colorable and $H-e$ is *n*-colorable, we conclude that there is an *n*-coloring *f* of $H-e$ such that $f(a)=f(a')$.

Therefore, for any two distinct colors i,j there is an *n*-coloring *f* of *Q* such that $f(a)=i$ and $f(b)=j$.

It remains to show that for any $\in>0$, if *f* is an $(n-\in)$-circular coloring of *Q*, then $f(a)\cap f(b)=\phi$.

This would follow if we can prove that $f(a)=f(a')$ because $f(b)\cap f(a')=\phi$ by definition.

Assume to the contrary that there is an $(n-\in)$-circular coloring *f* of *Q* such that $f(a)\neq f(a')$.
Recall that *f* maps each vertex *v* of *Q* to a unit length arc of a circle *C* of length $n-\in$.

Let p_0 be a point of *C* lies in the arc $f(a)-f(a')$. Starting from p_0, we put *n* points $p_0,p_1,\cdots,p_{n-1}$ on the circle *C* (consecutively along the clockwise direction) such that the distance from p_i and p_{i-1} is equal to $(n-\in)/n<1$.Since for any v, $f(v)$ is a unit length arc, so $f(v)$ contains at least one of the points p_i.Define an *n*-coloring *c* of *Q* as follows: $c(v)=i$ if and only if $p_i\in f(v)$ and $p_j\notin f(v)$ for any $j<i$.
This is indeed an *n*-coloring of *Q*, as every vertex of *Q* is colored by one of the *n* colors, and two adjacent vertices have distinct colors.
But $c(a)=0$ and $c(a')\neq 0$.
This means that *c* is indeed an *n*-coloring of *H*, contrary to our assumption that *H* has chromatic number $n+1$.
Hence the proof.

IV. PLANAR GRAPHS

Our next result concerns planar graphs. Vince [1]asked for families of planar graphs whose star chromatic numbers are strictly between 2 and 3.

It seems that such graphs are abundant (See [3, 4])while not many planar graphs are known to have star chromatic number exactly 3. Obviously if a 3-chromatic planar graph *G* contains a triangle, then $\chi^*(G)=3$. The first triangle free planar graph *G* with $\chi^*(G)=3$ was found by Gao (personal communication).

Denote by W_{2n+1} the graph obtained from the circuit C_{2n+1} by adding a vertex *v* and connecting *v* to every vertex of the circuit C_{2n+1}.

The graph W_{2n+1} is called the $(2n+1)$-wheel, and the edges connecting *v* to vertices of C_{2n+1} are called the spokes of the wheel.

Through an existensive check, Gao showed that the graph obtained from the 5-wheel by subdividing each of the five spokes by precisely one additional vertex has star-chromatic number 3.

We prove that for all integers *n*, the graph obtained from W_{2n+1} by subdividing its $2n+1$ spokes has star-chromatic number 3.

This is the first (non-trivial) infinite family of triangle free planar graphs with star-chromatic number 3.

Theorem:

Let G_{2n+1} be the graph obtained from W_{2n+1} by subdividing each of its $2n+1$ spokes by precisely one additional vertex. Then $\chi^*(G)=3$.

Proof:

We again use the definition of circular coloring in the proof let $V=\{v_1,x_0,x_1,\cdots,x_{2n}\}$ be the vertex set of W_{2n+1} while *v* is connected to all the $x_i^{'}$s and the set $\{x_0,x_1,\cdots,x_{2n}\}$ induces a circuit with edges (x_i,x_{i+1}).

The graph G_{2n+1} is obtained from W_{2n+1} by subdividing each edge (v,x_i) into two edges.

Let $u_i,\ i=0,1,\cdots,2n$ be the vertex which subdivides the edge (v,x_i).

It is easy to see that $\chi(G_{2n+1})=3$.

Therefore $\chi^*(G_{2n+1})\le 3$.

Suppose $\chi^*(G_{2n+1})=r<3$

Let $C:V(G_{2n+1})\to C^{(r)}$ be an *r*-circular coloring of G_{2n+1}. As *v* is adjacent to all the u_i's_i all the intervals $c(u_i)$ are disjoint from $c(v)$.

Since $r<3$ and $c(x_i)$ is disjoint from $c(u_i)$, we have $c(x_i)\cap c(v)\ne\phi$, for $i=0,1,\cdots,2n$.

For any $i\in\{0,1,\cdots,2n\}$ we cannot have $c(x_i)=c(v)$, for otherwise we would have $c(x_i)\cap c(x_{i+1})\ne\phi$. While x_i is adjacent to x_{i+1}.

Therefore $c(x_i)$ contains one of the end points of $c(v)$. Let p,q be the two end points of $c(v)$. Without loss of generality, We assume that $c(x_0)$ contains *p*.

Since $c(x_1)$ is distinct from $c(x_0),\ c(x_1)$ must contain *q*.

For the same reason, $c(x_2),c(x_3),\cdots,c(x_{2n})$ must alternately contain *p* and *q*.

Thus $p\in c(x_{2n})$ and hence $c(x_{2n})\cap c(x_0)\ne\phi$.

Contrary to our assumption that *c* is an *r*-circular, coloring of G_{2n+1}.

This completes the proof.

V. CRITICAL GRAPHS WITH LARGE GIRTH

A graph *G* is called $(m+1)$- critical if $\chi(G)=m+1$ and for any edge $e\in E(G),\ \chi(G-e)=m$.

We prove in this section that critical graphs *G* with large girth have star-chromatic numbers $\chi^*(G)$ close to $\chi(G)-1$.

Indeed we shall prove the following stronger statement.

Theorem:

Let $m\ge 2$ and $t\ge 1$ be integers. Let *G* be a graph. If *G* has a vertex *x* such that $G-x$ is *m*-colorable and any circuit of *G* containing *x* has length at least $m(t-1)+2$ then $\chi^*(G)\le m+\frac{1}{t}$.

Proof:

To prove this theorem, we shall use a characterization of the star-chromatic number of a graph given by [11].

First we need a definition.
Let *G* be a graph, and let *D* be an orientation of *G*. For a cycle *C* in *D*, denote by C^+ and C^- the sets of edges of *C* obtained 'forward' and 'backward' respectively, with respect to a sense of traversal of *C*(which we may suppose chosen so that $|C^+| \geq |C^{-1}|$).Set

$$f(C,D) = \frac{|C^+|}{|C^-|} + 1,\ f(D) = \max_C f(C,D)$$

It is shown in [11] that $\chi^*(G)$ is equal to the minimum of $f(D)$ over all orientations *D* of *G*.
We now proceed to prove theorem:
Let *G* be a graph and let *x* be a vertex of *G* such that $G-x$ is *m*-colorable, and any circuit of *G* containing *x* has length at least $m(t-1)+2$.
To prove that,

$\chi^*(G) \leq m + \frac{1}{t}$, it suffices to find an orientation *D* of *G* such that $f(D) \leq m + \frac{1}{t}$.

Let $\Delta : V(G) - \{x\} \to \{1,2,\cdots,m\}$ be an *m*-coloring of $G-x$.

Let *D* be the orientation of *G* in which (u,v) is an arc of *D* if and only if (u,v) is an edge of *G* and either $u = x$ or $\Delta(u) < \Delta(v)$.

Note that $D-x$ contains no directed path of length more than *m*-1.
Let *C* be a cycle of *D*.
If $x \notin C$ then $f(C,D) \leq m$ because *C* contains no directed path of length more than $m-1$.
If $x \in C$ then $C-x$ is a path *P* of length at least $m(t-1)$, and *P* contains no directed path of length more than $m-1$.
Let P^+ and P^- be the sets of forward and backward edges of *P* respectively. Then

$$|P^+| + |P^-| \geq m(t-1),\ |P^+| \leq (m-1)(|P^-|+1)$$

Hence,

$$f(C,D) = \frac{|P^+|+1}{|P^-|+1} + 1$$
$$\leq m + \frac{1}{|P^-|+1}$$
$$\leq m + \frac{1}{t}$$

Therefore $\chi^*(G) \leq f(D) \leq m + \frac{1}{t}$.
This completes the proof.

VI. CIRCULAR CHROMATIC INDEX OF GRAPH OF GIRTH

The circular chromatic index $\chi'(G)$ of *G* is defined to be the circular chromatic number of $L(G)$. It is not difficult to show that $\Delta \leq \chi_c'(G) \leq \Delta + 1$ for every graph *G* with maximum degree Δ.

Decompositions of Graphs:

In this we introduce the notions of decomposable graphs and graph decompositions which play a key role in our arguments.

Let *G* be a graph of maximum degree Δ. For an integer *k* a sub *k*-factor of *G* is any spanning subgraph of maximum degree at most *k*. Note that a 1-factor of graph is a perfect matching, while a sub-1-factor is just a matching.

An *l*-decomposition of a graph *G* is a decomposition of *G* into $[l/2]$ edge disjoint sub-2-factors, one of which is required to be a matching if *l* is odd.
If *G* has an *l*-decomposition, then it is said to be *l*-decomposable.

For our purposes, Δ-decomposable graphs of maximum degree Δ will be of interest. It is easy to find examples of graphs that are not of this type (Consider any cubic graph with no perfect matching). However, graphs of even maximum degree always have a Δ-decomposition.

Theorem:

A multigrpah*G* contains a 1-factor if and only if for each $S \subset V(G)$. The following holds,

$C_{odd}(G \setminus S) \leq |S|$.

Where, $C_{odd}(H)$ denotes the number of components of *H* of odd order.

Proof:

The proof of the following lemma follows ides used to prove the theorem of Peterson [5] on the existence of 1-factors in bridgeless cubic graphs.

Lemma:

Let *G* be a connected multigraph. Suppose that all the vertices of *G* have the same odd degree $\Delta \geq 3$, except possibly for one vertex *u* of degree at most Δ. If *G* does not contain any edge-cut of size less than Δ with a possible exception of the cut formed by all the edges incident with *u*, then *G* contains a matching that covers $V_\Delta(G)$.

Proof:

We distinguish two cases regarding the parity of the order of *G*.

Assume first that $|V(G)|$ is even.

We use theorem to show that *G* contains a 1-factor consider a subset $S \subset V(G)$.

If $S = \phi$, then $C_{odd}(G \setminus S) = 0$ as *G* is connected. Otherwise, the number of edges between the vertices of *S* and $V(G) \setminus S$ is atmost $\Delta |S|$. Since each vertex of *S* has degree at most Δ.

On the other hand, *G* contains at most one edge cut of size less than Δ and so all but at most one component of $G \setminus S$ are joined to the vertices of *S* by at least Δ edges. In particular,

$\Delta(C_{odd}(G \setminus S) - 1) < \Delta |S|$.

which implies $C_{odd}(G \setminus S) \leq |S|$ are required.

By theorem ,*G* has a 1-factor.

Assume now that $|V(G)|$ is odd.

Let δ be the degree of *u*.

Since Δ is odd, $\delta < \Delta$ by the hand-shaking lemma.

Thus, it is enough to show that the graph $G' = G \setminus \{u\}$ has a 1-factor.

Note that G' is connected. Otherwise, a proper subset of the edges incident with *u* would form an edge-cut in *G*, which was assumed not to be the case.

We again use theorem, in order to show the existence of a 1-factor in G'.

Let $S' \subset V(G')$ and set $S = S' \bigcup \{u\}$.

If $S' = \phi$, then $S = \{u\}$ and the graph $G \setminus S = G' \setminus S'$ is connected.

Since its order is even, the condition of theorem is satisfied.

If $S' = \phi$, then the number of edges between δ and $V(G) \setminus S$ is atmost $\Delta |S'| + \delta$.

Each component *C* of $G \setminus S$ is joined by at least Δ edges to the vertices of *S*.

Indeed, if not, then the edges incident with *C* from an edge-cut of size smaller than Δ.

By the assumption of the lemma, the edges forming this edge-cut must be exactly all the edges incident with the vertex *u*.

Then, the graph $G[C \bigcup \{u\}]$ is a component of the graph *G* but since *G* is connected, we infer that $S' = \phi$.

Hence, each component of $G \setminus S$ is joined by at least Δ edges to the vertices of *S*. Since there are at most $\Delta |S'| + \delta < \Delta(|S'| + 1)$ such edges, the graph $G \setminus S = G' \setminus S'$ consists of at most $|S'|$ components.

In particular, $C_{odd}(G' \setminus S') \leq |S'|$ and G' contains a 1-factor by theorem.

Hence the proof.

Lemma:

Every essentially Δ-edge-connected multigraph*G* of odd maximum degree $\Delta \geq 3$ has a matching which covers all the vertices of $V_\Delta(G)$.

Proof:

As long as *G* contains a pair of non-adjacent vertices *v* and v' whose degree sum up to at most Δ, identify v and v' (preserving multiple edges if they arise). Just like *g*, the resulting multigraph G' is essentially Δ-edge-connected.

If G' contains a pair of vertices *u* and *v* both of whose degrees arc smaller than Δ, add the edge *uv* to *G* (This preserves the essential edge-connectivity since the sum of the degrees of *u* and *v* is larger than Δ).
Repeat this process until there is no such pair of vertices and call the resulting multigraph G''. IF G'' is Δ-regular it is also Δ-edge-connected.
Otherwise, G'' contains exactly one vertex *z* of degree smaller than Δ and the only edge-cut in G'' of size less than Δ consists of the edges incident with *Z*.
In each case, lemma implies that G'' has a matching which covers $V_\Delta(G'')$. The matching consisting of the corresponding edges in *G* has the required property.Hence the proof.

Lemma:

Let *T* be a tree of maximum degree Δ and $X \subseteq V(T)$ set $T_\Delta = V_\Delta(T) \setminus X$. If no vertex of T_Δ has Δ neighbors in *X*, and atmost one vertex of T_Δ has $\Delta - 1$ neighbours in *X*, then there is a matching *M* in $T \setminus X$ covering all vertices of T_Δ.

Proof:

We can assume that T_Δ is non-empty for otherwise the claim holds for trivial reasons.

Let *r* be a vertex of T_Δ with $\Delta - 1$ neighbours in *X*, if there is one otherwise, choose *r* to be an arbitrary vertex of T_Δ orient the edges of the tree *T* away from the vertex *r*.

Let M_0 be a set of edges obtained by choosing, for each vertex $v \in T_\Delta$, one outgoing edge vw ending in a vertex $w \notin X$.

Such a choice can always be made, since if $v \neq r$, then atmost $\Delta - 2$ out of the $\Delta - 1$ outgoing edges end in *X* (and a similar argument applies to *r*).

Finally, we change M_0 into a matching *M* by removing certain edges. For every maximal directed path *P*in M_0, remove every second edge (starting with the second one from the beginning of *P*).
Clearly, *M* is a matching and since each *P* ends in a vertex not contained in T_Δ, *M* still covers all the vertices of T_Δ.

Hence the proof.

VII. CONCLUSION

We already know that the chromatic number of any planar graph, while it is known that the chromatic number of a planar graph is *NP*-complete, and also we present an infinite family of triangle – free planar graphs whose star chromatic number of color critical graphs.

REFERENCES

[1] A.Vince Star Chromatic number J.Graph theory 12(4): 551-559, 1988.

[2] D.R.Guichard, Acyclic graph coloring and the complexity of the Star Chromatic Number, J.Graph theory 17 (1993), 129-134.

[3] H.L.Abott and B.Zhou, The Star Chromatic number of a graph, J.Graph theory 17 (1993), 349-360.

[4] X.Zhu, Star Chromatic numbers and products of graphs, J.Graph Theory 16 (1992), 557-569

[5] T.N.Cusick and C.Pomerance, view-obstruction problems 111, J.Number theory 19 (1984), 131-139.

[6] XudingZhu.Circular Chromatic Number: a Survey Discrete Math, 229 (1-3) 371-410.

[7] J.A.Bondy and P.Hell, A note on the Star Chromatic number, J.Graph theory 14 (1990) 479-482.

[8] Z.Pan and X.Zhu, Density of the Circular chromatic number of Series-Parallel graphs, J. Graphs Theory 46 (2004), 57-68.

[9] O.V.Borodin, S.J.Kim, A.V.Kostahka and B.D.West Homomorphism from sparse graph with large girth, manuscript.

[10] B.Mohar, Choosability for the circular chromatic number. http://www.fmf.unilj.si/~mohar/problems/p0201/choosabilitycircular.html, 2003.

[11] L.A.Goddyn, M.Tarsi and C.Q.Zhang, on (*k*,*d*)-colorings and fractional nowhere-zero flows, J.Graph theory to appear.

[12] W.Deuber and X.Zhu, Circular coloring of weighted graphs, manuscript, 1994.

[13] N.Fountoulakis, C.McDiarmid and B.Mohar, Brooks theorem for digraphs in preparation.