On Cubic Harmonious Graphs

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Abstract-*A (n,m) graph G=(V,E) is said to be Cubic Harmonious Graph(CHG) if there exists an injective function f:V(G)→{1,2,3,………m3+1} such that the induced mapping f * chg : E(G)→ {13,23,33,……….m3} defined by f*chg(uv) = (f(u)+f(v)) mod (m3+1) is a bijection. We have proved that the Hn graph, P_n @ P_m , pK_(1,q) ∪ rK_(1,S) , U_(r=1)^t K_(1,r) , < K1,a, K1,b, K1,c, K1,d> are cubic harmonious.*

Keywords-Cubic harmonious labeling. Cubic harmonious graph, Graceful graph, H-graph , Star graph

I. INTRODUCTION

A *(n,m)*- graph G is to be harmonious, if there is an injective function $f : V(G) \rightarrow Z_m$,where Z_m is the group of integers modulo *m*, such that the induced function $f^* : E(G) \rightarrow Z_q$, defined by $f^*(uv) = f(u) + f(v)$ for each edge $uv \in E(G)$ is a bijection.All the graphs considered here are finite, connected and undirected with no loops and multiple edges. A detailed survey of graph labeling can be found in [2]. Square harmonious graphs were introduced in [8]. Cubic graceful graphs were introduced in [4]. Cubic harmonious graphs were defined in [5].

Definition 1
The ***Trivial graph*** K_1 or P_1 is the graph with one vertex and no edges

Definition 2
The ***H-graph*** of a path P_n is the graph obtained from two copies of P_n with vertices $v_1, v_2, \ldots v_n$ and $u_1, u_2, \ldots u_n$ by joining the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n+1}{2}}$ if n is odd, and the vertices $v_{\frac{n+1}{2}}$ and $u_{\frac{n}{2}}$ if n is even.

Definition 3
A complete bipartite graph $K_{1,n}$ is called a ***star*** and it has (n+1) vertices and n edges

II. MAIN RESULTS

Theorem 2.1
The H_n*-graph* G is cubic harmonious for all $n > 2$.
Proof:

Let $V(G) = \{u_i , v_i\}$; $1 \le r \le n$
And

$$E(G) = \begin{cases} u_i u_{i+1} \\ v_i v_{i+1} & \text{if } 1 \le i \le n-1 \\ u_{\frac{n+1}{2}} v_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ u_{\frac{n+1}{2}} v_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}$$

Then $|V(G)| = 2n$ and $|E(G)| = 2n-1$
Let $f: V(G) \rightarrow \{1,2,\ldots,(2n-1)^3+1\}$ be defined as follows:

<u>*Case (i) when n is odd*</u>

$f(u_1) = (2n-1)^3+1$

$f(u_i) = (2n+1-i)^3 + (2n-1)^3 + 1 - f(u_{i-1})$; $2 \le i \le n$

$$f(v_{\frac{n+1}{2}}) = 9n^3 - 12n^2 + 6n - f\left(u_{\frac{n+1}{2}}\right)$$

$$f\left(v_{\frac{n+1}{2}-j}\right) = \left(\frac{n+1}{2} + j - 1\right)^3 + (2n-1)^3 + 1 - f\left(v_{\frac{n+1}{2}-j+1}\right); 1 \le j \le \frac{n-1}{2}$$

$$f\left(v_{\frac{n+1}{2}+j}\right) = \left(\frac{n+1}{2} - j\right)^3 + (2n-1)^3 + 1 - f\left(v_{\frac{n+1}{2}+j-1}\right); 1 \le j \le \frac{n-1}{2}$$

<u>*Case (ii) when n is even*</u>

$f(u_1) = (2n-1)^3+1$

$f(u_i) = (2n+1-i)^3 + 8n^3 - 12n^2 + 6n - f(u_{i-1})$; $2 \le i \le n$

$$f(v_{\frac{n}{2}}) = 9n^3 - 12n^2 + 6n - f\left(u_{\frac{n+1}{2}}\right)$$

$$f\left(v_{\frac{n}{2}-i}\right) = \left(\frac{n}{2} + i\right)^3 + 8n^3 - 12n^2 + 6n - f\left(v_{\frac{n}{2}-i+1}\right); 1 \le i \le \frac{n}{2} - 1$$

$$f\left(v_{\frac{n}{2}+i}\right) = \left(\frac{n}{2} + 1 - j\right)^3 + 8n^3 - 12n^2 + 6n - f\left(v_{\frac{n}{2}+i-1}\right); 1 \le j \le \frac{n}{2}$$

Let f^* be the induced edge labeling of f .

Then

$$f^*(u_i u_{i+1}) = (2n-i)^3 \quad ; \quad 1 \le i \le n-1$$

$$f^*\left(u_{\frac{n+1}{2}} v_{\frac{n+1}{2}}\right) = n^3 \; ; \quad if \;\; n \;\; is \;\; odd$$

$f^*\,(u_{\frac{n}{2}+1}\, v_{\frac{n}{2}}) = n^3$; *if n is even.*

$f^*(v_i v_{i+1}) = (n-i)^3 \quad ; \quad 1 \le i \le n-1$

The induced edge labels are distinct and they are *{1^3,2^3,……….(2n-1)^3}.*

Hence the theorem.

Theorem 2.2

The graph $P_n @ P_m$ is cubic harmonious for all $n,m \ge 2$.

Proof:

Let P_n be the path with n vertices and P_m be the path with m vertices. The graph $P_n @ P_m$ is obtained by one pendant vertex of the i^{th} copy of P_n with i^{th} vertex of P_m we denote this graph by G.

Let $V(G) = u_i \;;\quad 1 \le i \le mn$

And

$E(G) = \begin{cases} u_i u_{i+1}\,; & 1 \le i \le n-1 \\ u_i u_j\,; & 1 \le j \le n,\; nj+1 \le i \le m(n-1)+2 \\ u_i u_j\,; & 1 \le k \le n,\; 1 \le j \le n-1,\; ((m-1)(k-1)+(n+1)+j \le i \le (k+1)n \end{cases}$ |V(G)| = mn and |E(G)| = $mn-1$.

Let $f: V(G) \to \{1,2,\ldots,(mn-1)^3+1\}$ be defined as follows:

$$f\,(u_{11}) = (mn-1)^3+1$$

$$f\,(u_i) = (mn-i+1)^3 + mn^3 - 3(mn)^2 + 3mn - f(u_{i-1});\; 2 \le i \le n$$

$$f(u_i) = (mn-i+1)^3 + (mn)^3 - 3(mn)^2 + 3mn - f(u_j);\; 1 \le j \le n,\; (nj+1)+(j-1)(m-1) \le i \le m(n-1)+2$$

$$f(u_i) = (mn-i+1)^3 + (mn)^3 - 3(mn)^2 + 3mn - f(u_{i-1});\; 1 \le k \le n,\; 1 \le j \le m-2,\; (m-1)(k-1)+(n+1)+j \le i \le (k+1)n$$

Let f^* be the induced edge labeling of f .Then

$f^*(u_i u_{i+1}) = (mn-i)^3 \;; \quad 1 \le i \le n-1$

$f^*(u_i u_j) = (mn-i+1)^3; \quad 1 \le r \le n,\, nj+1 \le i \le m(n-1)+2$

$f^*(u_i u_{i-1}) = (mn-i+1)^3; \quad 1 \le k \le n,\; 1 \le j \le m-2,\; (m-1)(k-1)+(n+1)+j \le i \le (k+1)n$

The induced edge labels are distinct and cubic .They are *{1^3,2^3,………..(mn-1)^3}.*

Hence the theorem

Theorem 2.3

Let G be the graph obtained from two copies of P_n with vertices $u_1, u_2, \ldots u_n$ and $v_1, v_2, v_3, \ldots, v_n$ by joining the fixed vertices u_r and v_r by means of an edge, for $1 \le r \le n$. Then the graph $(P_n: G)$ is cubic harmonious.

Proof: Let G denote the graph $(P_n: G)$

Let $V(G) = \begin{cases} u_i\,; & i = 1,2,3\ldots n \\ v_i\,; & i = 1,2,3\ldots n \end{cases}$

and $E(G) = \begin{cases} u_i u_{i+1}; & i = 1,2,3\ldots(n-1) \\ v_i v_{i+1}; & i = 1,2,3\ldots(n-1) \\ u_r v_r\,; & \end{cases}$

where `$r$` is a fixed vertex,for $1 \le r \le n$

So, |V(G)| = $2n$ and |E(G)| = $2n-1$.

Define $f: V(G) \to \{1,2,\ldots\ldots,(2n+1)^3+1\}$ by

The induced edge mapping are

$$f\,(u_1) = (2n-1)^3+1$$

$f\,(u_i) = (2n-i+1)^3 + 8n^3 - 12n^2 + 6n - f(u_{i-1})\;;\; 2 \le i \le n$

$f(v_{\frac{n}{2}}) = 9n^3 - 12n^2 + 6n - f\,(u_r)$ where ‘r’ is the fixed vertices for $1 \le r \le n$

$$f\left(v_{r-j}\right) = (n-r+j)^3 + 8n^3 - 12n^2 + 6n - f\left(v_{r-j+1}\right);\; 1 \le j \le r-1$$

$$f\left(v_{r+j}\right) = (n-r-j+1)^3 + 8n^3 - 12n^2 + 6n - f\left(v_{r+j-1}\right);\; 1 \le j \le n-r$$

The induced edge mapping are

$f^*\,(u_i u_{i+1}+1) = (2n-i)^3\;; \quad 1 \le i \le n$

$f^*(u_r v_r) = n^3$ where ‘r’ is a fixed vertex for $1 \le r \le n$

$$f^*(v_i v_{i+1}) = (n-1)^3; 1 \le r \le n$$

The vertex labels are in the set $\{1,2,\ldots\ldots\ldots,(2n-1)^3+1\}$. Then the edge labels are arranged in the set $\{1^3,2^3,3^3\ldots.(2n-1)^3\}$ So the vertex labels are distinct and the edge labels are also cubic and distinct . So the graph (P_n:G)) is a cubic harmonious.

Theorem 2. 4

The graph $pK_{1,q} \cup rK_{1,S}$ is cubic harmonious for $p,q,r,s \geq 1$

Proof. Let G denote the graph $pK_{1,q} \cup r K_{1,S}$

Let V(G) = u_m : $1 \leq m \leq p$;
u_{mn} : $1 \leq m \leq p$; $1 \leq n \leq q$
v_t : $1 \leq t \leq r$;
v_{tw} : $1 \leq t \leq r$; $1 \leq w \leq s$
and E(G) = $u_m u_{mn}$: $1 \leq m \leq p$; $1 \leq n \leq q$
$v_t v_{tm}$: $1 \leq t \leq r$; $1 \leq w \leq s$
Then $|n(G)| = p(1+q) + r(1+s)$ and
$|m(G)| = pq + rs$

Define $f: V(G) \rightarrow \{1,2 \ldots\ldots, (pq+rs)^3+1\}$ by
$f(u_m) = (pq + rs - q(m-1))^3 + 1$; $1 \leq m \leq p$
$f(u_{mn}) = (pq + rs - (m-1)q - (n-1))^3$; $1 \leq m \leq p, 1 \leq n \leq q$
$f(v_t) = [s(r-t+1)]^3 + 1$; $1 \leq t \leq r$
$f(v_{tw}) = [s(r-t+1) - 1 - (w-1)]^3 + v_t$; $1 \leq t \leq r, 1 \leq w \leq s$

The induced edge mapping are
$f^*(u_m u_{mn}) = [pq + rs - (m-1)q - (n-1)]^3$; $1 \leq m \leq p$ $1 \leq n \leq q$;
$f^*(v_t v_{tm}) = [s(r-t+1) - (w-1)]^3$; $1 \leq t \leq r$; $1 \leq w \leq s$;

The vertex labels are arranged in the set $\{1,2,\ldots\ldots\ldots,(pq + rs)^3 + 1\}$. Then the edge labels are arranged in the set $\{1^3, 2^3, \ldots\ldots (pq+rs)^3\}$. Therefore the edge label are distinct and vertex labels are also distinct. So G is harmonious.

Theorem 2.5 `

The graph $U_{r=1}^{t} K_{1,r}$ is a cubic harmonious for all t ≥ 1

Proof : Let *G* denote the graph $U_{r=1}^{t} K_{1,r}$

Let *V (G)* = v_r; for *r =1,2...........,t*
v_{rs} for *r = 1,2........,t* ; *s= 1,2.........r*
and
E (G) =$v_r v_{rs}$ for *r = 1,2........,t ; s= 1,2.........r*

So $n(G) = \frac{t(t+3)}{2}$ and $|m(G)| = \frac{t^2+t}{2}$

Define f:V(G) $\longrightarrow \left\{1,2 \ldots\ldots\ldots, \left(\frac{t^2+t}{2}\right)^3 + 1\right\}$ by

$f(v_r) = \left(\left(\frac{t^2+t}{2}\right) - [(r-1) + (r-2) + \cdots \ldots\ldots\ldots + 1]\right)^3 + 1$; $1 \leq r \leq t$

$f(v_{rs}) = \left[\left(\frac{t^2+t}{2}\right) - [(r-1) + (r-2) + \cdots \ldots\ldots\ldots + 1] - (s-1)\right]^3$; $1 \leq r \leq t$, $1 \leq s \leq r$ The induced edge mapping are

$f^*(v_r v_{rs}) = \left[\left(\frac{t^2+t}{2}\right) - [(r-1) + (r-2) + \cdots \ldots + 1] - (s-1)\right]^3$; $1 \leq r \leq t$, $1 \leq s \leq r$

The vertex labels are in the set $\left\{1,2 \ldots\ldots, \left(\frac{t^2+t}{2}\right) + 1^3\right\}$. Then the edge label arranged in the set $\left\{1^3, 2^3, 3^3 \ldots\ldots . \left(\frac{t^2+t}{2}\right)^3\right\}$. So the vertex labels are distinct and the edge labels are also distinct and cubic. So the graph $U_{r=1}^{t} K_{1,r}$ is a cubic harmonious for all t ≥ 1.

Theorem 2.6

Let G be the graph < $K_{1,a}$, $K_{1,b}$, $K_{1,c}$, $K_{1,d}$> obtained by joining the middle vertices of $K_{1,a}$, $K_{1,b}$, $K_{1,c}$, $K_{1,d}$ to another vertex 'u' is cubic harmonious for all a ,b ,c, d ≥ 1

Proof:

Let G be a combination of the star graph < $K_{1,a}$, $K_{1,b}$, $K_{1,c}$, $K_{1,d}$>

Let V (G) = { $u, u_n, u_{1p}, u_{2q}, u_{3r} u_{4s}$; $1 \leq n \leq 4$, $1 \leq p \leq a$, $1 \leq q \leq b$, $1 \leq r \leq c$, $1 \leq s \leq d$ }

and E (G) = uu_n ; $1 \leq n \leq 4$
u_1u_{1p} ; $1 \leq p \leq a$
u_2u_{2q} ; $1 \leq q \leq b$
u_3u_{3r} ; $1 \leq r \leq c$
u_4u_{4s} ; $1 \leq s \leq d$

So, $|n(G)| = a+b+c+d+5$
$|m(G)| = a+b+c+d+4$

Define f: V (G) $\rightarrow \{1,2 \ldots\ldots\ldots\ldots, (a+b+c+d+4)^2\}$ by
$f(u) = (m(G))^3 + 1$;
$f(u_n) = (n(G) - n)^3$; $1 \leq n \leq 4$
$f(u_{1p}) = (a + b + c + d - (p-1))^3 + (m(G))^3 + 1 - f(u_1)$; $1 \leq p \leq a$
$f(u_{2q}) = (b + c + d - (q-1))^3 + (m(G))^3 + 1 - f(u_2)$; $1 \leq q \leq b$

$f(u_{3r}) = \left(c + d - (r-1)\right)^3 + (m(G))^3 + 1 - f(u_3);\quad 1 \leq r \leq c$

$f(u_{4s}) = \left(d - (s-1)\right)^3 + (m(G))^3 + 1 - f(u_4);\quad 1 \leq s \leq d$

The induced edge mapping are

$f^*(uu_n) = \left(n(G) - n\right)^3;\quad 1 \leq n \leq 4$

$f^*(u_1u_{1p}) = (a + b + c + d + 1 - p)^3;\quad 1 \leq p \leq a$

$f^*(u_2u_{2q}) = (b + c + d + 1 - q)^3;\quad 1 \leq q \leq b$

$f^*(u_3u_{3r}) = (c + d + 1 - r)^3;\quad 1 \leq r \leq c$

$f^*(u_4u_{4s}) = (d + 1 - s)^3;\quad 1 \leq s \leq d$

The vertex labels are in the set $\{1, 2 \ldots\ldots, (a + b + c + d + 4)^3 + 1\}$ Then the edge label arranged in the set $\{1^3, 2^3, 3^3 \ldots\ldots\ldots (a + b + c + d + 4)^3\}$. So the vertex labels are distinct and edge labels are also distinct and cubic. So the graph G is cubic harmonious for all $a, b, c, d \geq 1$.

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