Characterization Of Contra Weakly Generalized Closed Graph Functions On Separation Axioms

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Abstract- *The purpose of this paper is to define and study various types of graph functions such as contra weakly generalized closed graphs, Almost contra weakly generalized closed graph and locally contra weakly generalized closed graphs in Topological spaces. And also, we defined some new functions in order to characterize these graphs by utilizing the notion of weakly generalized closed sets.*
AMS Subject Classification: 54C10.

Keywords: contra-wg-closed graphs, almost contra wg- closed graphs and locally contra wg-closed graph.

I. INTRODUCTION

Various generalizations of graph functions have been introduced and investigated in topological spaces. One of the most significant of those notations is locally contra closed graph [3] which was introduced by Baker et al. Likewise, In 2007 C. W. Baker was investigated contra closed graph [2] and regular closed graph [2]. Based on that, new class of closed graph defined, such as Contra g-closed graph[4], G-regular graph[9] and strongly contra g-closed graph[9] by Caldas et al., and Keskin et al. in 2007 and 2009 respectively.

The notions of continuous function have been found to be applied in many fields. Dontchev [5] introduced the notion of contra-continuity in 1996. Recently, few contra-continuous functions, such as contra-g-continuous [4], almost contra-g-continuous[9], contra-b-continuous [1] almost contra pre continuous [6] and contra pre continuous[8] were introduced and analysed.

The purpose of this paper is to define and study various types of closed graphs such as contra weakly generalized closed graph, Almost contra weakly generalized closed graph and locally contra weakly generalized closed graphs in Topological spaces. Some more properties of functions with these different types of contra closed graphs are investigated. And also, we defined some new continuous functions in order to characterize these graphs by utilizing the notion of weakly generalized closed sets.

Throughout the paper, the spaces (X, τ) and (Y, σ)(or simply X and Y) stand for topological spaces. Let A be a subset of X. The closure of A and the interior of A will denoted by Cl(A) and Int(A), respectively. The family of open and closed sets of X will be denoted by O(X) and C(X).

II. PRELIMINARIES

In this section, we list some definitions which are used in this sequel.

Definition: 2.1 A subset A of a space (X, τ) is said to be

i. generalized closed (g-closed) sets [10] if $Cl(A) \subset U$ whenever $A \subset U$ and U is open in X.
ii. weakly generalized closed (wg-closed) sets [11] if $Cl(Int(A)) \subset U$ whenever $A \subset U$ and U is open in X.

The complement of g-closed set (resp. wg-closed set) is said to be g-open set (resp. wg-open set). The family of all g-open sets (resp. wg-open set) is denoted by GO(X) (resp. WGO(X)). We set $GO(X, x) = \{V \in GO(X) / x \in V\}$ for $x \in X$. We define similarly $WGO(X, x) = \{V \in WGO(X) / x \in V\}$ for $x \in X$. The g-closure (resp. wg-closure) of a subset A of X is, denoted by g-Cl(A) (resp. wg-Cl(A)) [13], defined to be the intersection of all g-closed (resp. wg-closed) sets containing A.

***Definition**: 2.2* A subset A of a space (X, τ) is said to be

i. regular open[12] if A = Int(Cl(A).
ii. preopen [8] if $A \subset Int(Cl(A))$.

The family of all regular open (resp. preopen) sets in a space (X, τ) is denoted by RO(X) (resp. PO(X)). The complement of a regular open (resp. preopen) set is said to be regular closed (resp. preclosed) set.

Definition: 2.3 A function $f: (X, \tau) \to (Y, \sigma)$ is called

i. contra-continuous [5] if $f^{-1}(V)$ is closed in X for every open set V in Y.
ii. almost continuous [12] if $f^{-1}(V)$ is open in X for each regular open set V of Y.
iii. contra-g-continuous [4] if $f^{-1}(V)$ is g-closed in X for every open set V in Y.

iv. contra-wg-continuous if $f^{-1}(V)$ is wg-closed in X for every open set V in Y.

v. almost contra g-continuous [9] if $f^{-1}(V)$ is g-closed in X for every regular open set V in Y.

Definition: 2.4 A space (X, τ) is called

i. weakly hausdroff space [9] if each element of X is an intersection of regular closed sets.

ii. Hausdroff space (briefly., T_2 space) [4] if every pair of distinct points x, y ∈ X there exists U ⊆ O(X, x) and V ⊆ O(X, y) such that U ∩ V = ∅.

iii. Urysohn space (briefly., T_2' space) [4] if every pair of distinct points x, y ∈ X there exists U ⊆ O(X, x) and V ⊆ O(X, y) such that Cl(U) ∩ Cl(V) = ∅.

iv. S-closed [7] if every regular closed cover of X has a finite subcover.

Definition: 2.5 A space (X, τ) is called

i. wg-T_1 [14] if for every pair of distinct points x, y in X there exists a wg-open set U ⊂ X containing x but not y and a wg-open set V ⊂ X containing y but not x.

ii. wg-T_2 [14] if for every pair of distinct points x, y in X there exists disjoint wg-open sets U ⊂ X and V ⊂ X containing x and y respectively.

iii. wg-Compact [13] if every wg-open cover of X admits a finite subcover.

iv. wg-connected [13] if X is not the union of two disjoint nonempty wg-open sets .

Definition: 2.6 [4] Let f: $(X, \tau) \to (Y, \sigma)$ be any function. Then the subset $\{(x, f(x)) / x \in X\}$ of the product space $(X \times Y, \tau \times \sigma)$ is called the graph of f and is denoted by G(f).

Definition: 2.7 A graph G(f) of a function f: $(X, \tau) \to (Y, \sigma)$ is said to be locally contra closed [3] (resp. contra closed[2], regular closed[2], contra-g-closed[4] and G-regular[9]) graph if for each (x, y) ∈ (X $\times$ Y) - G(f), there exist U ∈ C(X, x) (resp. C(X, x), C(X, x), GO(X, x) and GC(X, x)) and V ∈ O(Y, y) (resp. C(Y, y), RC(Y, y), C(Y, y) and RO(Y, y)) such that (U $\times$ V) ∩ G(f) = Φ.

Lemma: 2.8 A graph G(f) of a function f: $(X, \tau) \to (Y, \sigma)$ is a locally contra closed (resp. contra closed, regular closed, contra-g-closed and G-regular) graph if for each (x, y) ∈ (X $\times$ Y) - G(f), there exist U ∈ C(X, x) (resp. C(X, x), C(X, x), GO(X, x) and GC(X, x)) and V ∈ O(Y, y) (resp. C(Y, y), RC(Y, y), C(Y, y) and RO(Y, y)) such that f(U) ∩ V = Φ.

III. CONTRA-WG-CLOSED GRAPH

In this section, we introduced contra-wg-closed graph in topological space. We defined contra-wg-continuous functions in order to characterize contra-wg-closed graphs by utilizing the notion of Urysohn and WG-closed space.

Definition: 3.1 A graph G(f) of a function f: $(X, \tau) \to (Y, \sigma)$ is said to be contra-wg-closed if for each (x, y) ∈ (X $\times$ Y) - G(f), there exist U ∈ WGO(X, x) and V ∈ C(Y, y) such that (U $\times$ V) ∩ G(f) = Φ.

Lemma: 3.2 A graph G(f) of a function f: $(X, \tau) \to (Y, \sigma)$ is a contra-wg-closed if for each (x, y) ∈ (X $\times$ Y) - G(f), there exist U ∈ WGO(X, x) and V ∈ C(Y, y) such that f(U) ∩ V = Φ.

Proof: We shall prove that f(U) ∩ V = Φ if and only if (U $\times$ V) ∩ G(f) = Φ. Let (U $\times$ V) ∩ G(f) ≠ Φ, then there exist (x, y) ∈ (U $\times$ V) and (x, y) ∈ G(f), so x ∈ U , y ∈ V and y = f(x) ∈ V, hence f(U) ∩ V ≠ Φ. Conversely, if f(U) ∩ V ≠ Φ, so there exist y ∈ f(U) and y ∈ V. Therefore, (x, y) ∈ U $\times$ V and y = f(x) implies that (x, y) ∈ G(f), then (U $\times$ V) ∩ G(f) ≠ Φ. Hence the result follows.

Theorem: 3.3 if a function f: $(X, \tau) \to (Y, \sigma)$ is contra-wg-continuous and Y is Urysohn, then G(f) is contra-wg-closed in X $\times$ Y.

Proof: Let (x, y) ∈ (X $\times$ Y) - G(f), then y ≠ f(x) and since Y is Urysohn, there exist open sets V and W in Y such that f(x) ∈ V and y ∈ W and Cl(V) ∩ Cl(W) = Φ. Since f is contra-wg-continuous, there exist an wg-open set U in X contains x such that f(U) ⊆ cl(V) which implies that f(U) ∩ Cl(W) = Φ. Hence by the Lemma 3.2, G(f) is contra-wg-closed in X $\times$ Y.

Theorem: 3.4 if the functions f: $(X, \tau) \to (Y, \sigma)$ and g: $(X, \tau) \to (Y, \sigma)$ are contra-wg-continuous functions, where Y is Urysohn space, then D = {x ∈ X: f(x) = g(x)} is wg-closed in X.

Proof: Let x ∈ (X - D), then f(x) ≠ g(x). Since Y is Urysohn, there exist open sets U and V such that f(x) ∈ U and g(x) ∈ V with Cl(U) ∩ Cl(V) = Φ. Again, since f and g are contra-wg-continuous functions, then $f^{-1}(Cl(U))$ and $g^{-1}(Cl(V))$ are wg-open sets in X. Let M = f^{-1} (Cl(U)) and N = $g^{-1}(Cl(V))$, then M and N are wg-open sets of X containing x. Let H = M ∩ N, then H is wg-open in X. Hence, f(H) ∩ g(H) = (f(M ∩ N) ∩ g(M ∩ N)) ⊆ (f(M) ∩ g(N)) ⊆ Cl(U) ∩ Cl(V) = Φ. Therefore, D ∩ H = Φ where H is wg-open therefore x ∉ wg-Cl(D). Thus D is wg-closed in X.

Theorem: 3.5 Let f: X → Y be a function and g: X → X × Y the graph function, given by g(x) = (x, f(x)) for every x ∈ X. Then g is contra-wg-continuous if f is contra-wg-continuous.

Proof: Suppose that g is contra-wg-continuous. Let U be an open subset of Y, then X $\times$ U is an open set of X × Y. Since g is contra-wg-continuous, we get that $g^{-1}(X \times U)$ is a wg-closed set in X. But $g^{-1}(X \times U) = \{x \in X: (x, f(x)) \in X \times U\} = \{x \in X: f(x) \in U\} = f^{-1}(U)$. Therefore, $f^{-1}(U)$ is wg-closed in X. Hence f is contra-wg-continuous.

Definition: 3.6 A subset A of a topological space X is said to be wg-dense in X if wg-Cl(A) = X.

Theorem: 3.7 Let f: $(X,\tau) \to (Y,\sigma)$ is contra wg-continuous and f: g: $(X,\tau) \to (Y,\sigma)$ is contra continuous. If Y is Urysohn, and f = g on wg-dense set $A \subseteq X$, then f = g on X.
Proof: Since f is contra wg-continuous, g contra continuous functions and Y is Urysohn by Theorem 3.4 D = { $x \in X$: f(x) = g(x)} is wg-closed in X. We have f = g on wg-dense set $A \subseteq X$. Since $A \subseteq D$ and A is wg-dense set in X, then X = wg-Cl(A) $\subseteq$ wg-Cl(D) = D. Hence f = g on X.

Theorem: 3.8 if (X,τ) is a topological space and f: $(X,\tau) \to (Y,\sigma)$ has a contra wg-closed graph, then the inverse image of a wg- closed set A of Y is wg-closed in X.
Proof: Assume that A is a wg-closed set of Y and $x \notin f^{-1}(A)$. For each $a \in A$, $(x, a) \notin G(f)$. by the Lemma 3.2, there exists a $U_a \in WGO(X, x)$ and $V_a \in C(Y, a)$ such that $f(U_a) \cap V_a = \Phi$. Since $\{A \cap V_a: a \in A\}$ is a wg-closed cover of the subspace A, since A is wg-closed, then there exists finite subset $A_0 \subset A$ such that $A \subseteq \cup\{V_a : a \in A_0\}$. Set $U = \cap\{U_a : a \in A\}$, but (X,τ) is s topological space, then $U \in WGO(X, x)$ and $f(U) \cap A \subseteq f(U_a) \cap \{\cup(V_a: a \in A_0\} = \Phi$. Therefore $U \cap f^{-1}(A) = \Phi$ and hence $x \notin$ wg-cl($f^{-1}(A)$). This shows that $f^{-1}(A)$ is wg-closed.

Definition: 3.9 A space X is said to be WG-Closed space if every wg-closed cover of X has a finite subcover.

Theorem: 3.10 let Y be a WG- Closed space. If (X,τ) is a topological space and a function f : X → Y has a contra-wg-closed graph, then f is contra wg-continuous.
Proof: Suppose that Y is WG- Closed space and G(f) is contra wg-closed. First we show that an open set of Y is wg-closed. Let U be an open set of Y and $\{V_i: i \in I\}$ be a a cover of U by closed sets Vi of U. For each $i \in I$, there exists a wg-closed set K_i of X such that $V_i = K_i \cap U$. then the family $\{K_i: i \in I\} \cup (Y-U)$ is a wg-closed cover of Y. Since Y is WG- Closed space, there exists a finite subset $I_0 \in I$ such that $Y = \cup\{K_i: i \in I_0\} \cup (Y - U)$. Therefore we obtain $U = \cup\{V_i: i \in I_0\}$. This shows that U is wg-closed. By the Theorem 3.8, $f^{-1}(U)$ is wg-closed in X for every open U in Y. Therefore, f is contra-wg-continuous.

IV. ALMOST CONTRA WG-CLOSED GRAPH

In this section, we introduced almost contra-wg-closed graph in topological space. We defined almost contra-wg-continuous functions in order to characterize almost contra-wg-closed graphs by utilizing the notion of some separation axiom, WG-Compact and WG-connected space.

Definition: 4.1 A graph G(f) of a function f: $(X,\tau) \to (Y,\sigma)$ is an almost contra-wg-closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in WGC(X, x)$ and $V \in RO(Y, y)$ such that $(U \times V) \cap G(f) = \Phi$.

Lemma: 4.2 A graph G(f) of a function f: $(X,\tau) \to (Y,\sigma)$ is an almost contra-wg-closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in WGC(X, x)$ and $V \in RO(Y, y)$ such that $f(U) \cap V = \Phi$.
Proof is an immediate consequence of Definition and the fact that for any subsets $U \subset X$ and $V \subset Y$, $(U \times V) \cap G(f) = \Phi$ if and only if $f(U) \cap V = \Phi$.

Definition: 4.3 A function f: $(X,\tau) \to (Y,\sigma)$ is said to be almost contra wg-continuous if
$f^{-1}(V) \in WGC(X)$, for every $V \in RO(X)$.

Theorem: 4.4 If f: $(X,\tau) \to (Y,\sigma)$ is almost contra wg-continuous and Y is T_2 space, then G(f) is almost contra wg-closed in $X \times Y$.
Proof: Let $(x, y) \in (X \times Y) - G(f)$, then $y \neq f(x)$. Since Y is T_2 space, there exist regular open sets V and W in Y such that $f(x) \in V$ and $y \in W$ and $V \cap W = \Phi$. Since f is almost contra wg-continuous, $f^{-1}(V)$ is wg-closed set in X containing x. if we take $U = f^{-1}(V)$, we have $f(U) \subset V$. Therefore, $f(U) \cap W = \Phi$. Hence G(f) is almost contra wg closed graph.

Theorem: 4.5 If f: X → Y is a injection with almost contra-wg-closed graph, then X is wg-T_1 space.
Proof: Let x and y be any two distinct points of X. Then, we have $(x, f(y)) \in (X \times Y) - G(f)$. Since G(f) is almost contra-wg-closed graph, there exist a wg-closed set U of X and regular open set V of Y such that $(x, f(y)) \in U \times V$ and $f(U) \cap V = \Phi$. Therefore we have $U \cap f^{-1}(V) = \Phi$ and $y \notin U$. Thus, $y \in (X - U)$ and $x \notin (X - U)$. We obtain $(X-U) \in WGO(X)$. This implies that X is wg-T_1 space.

Theorem: 4.6 If f: X → Y is a surjection with almost contra-wg-closed graph, then Y is weakly Hausdorff.
Proof: let y_1 and y_2 be any distinct points of Y. Since f is surjective, $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y)-G(f)$. By lemma 4.2, there exist a wg-closed set U of X and regular open set F of Y such that $(x, y_2) \in (U \times F)$ and $f(U) \cap F = \Phi$. Hence $y_1 \notin F$, then $y_2 \notin (Y-F) \in RC(Y)$ and $y_1 \in (Y-F)$. This implies that Y is weakly Hausdorff.

Theorem: 4.7 Let f: X → Y be an almost contra wg-continuous surjection. If X is WG-Compact, then Y is S-closed.

Proof: Let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y. Since f is almost contra wg-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a wg-open cover of X and hence there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$ therefore we have $Y = \cup\{V_\alpha : \alpha \in I_0\}$ and Y is WG-closed.

Theorem: 4.8 If f : X → Y is almost contra-wg-continuous surjection and X is wg-connected, then Y is connected.

Proof: Suppose that Y is not connected space. There exist nonempty disjoint open sets
V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen in Y. Since f is almost contra-wg-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are wg-open in X. Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are non empty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that X is not wg - connected. This contradicts that Y is not connected assumed. Hence, Y is connected.

V. LOCALLY CONTRA-WG-CLOSED GRAPHS

In this section, we define and investigate some fundamental properties of locally contra-wg-closed graph.

Definition: 5.1 A graph G(f) of a function f: $(X,\tau) \to (Y,\sigma)$ is said to be locally contra-wg-closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in WGC(X, x)$ and $V \in WGO(Y, y)$ such that $(U \times V) \cap G(f) = \Phi$.

Lemma: 5.2 A graph G(f) of a function f: $(X,\tau) \to (Y,\sigma)$ is a locally contra-wg-closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in WGC(X, x)$ and $V \in WGO(Y, y)$ such that $f(U) \cap V = \Phi$.

Proof: I is an immediate consequence of Definition and the fact that for any subsets $U \subset X$ and $V \subset Y$, $(U \times V) \cap G(f) = \Phi$ if and only if $f(U) \cap V = \Phi$.

Definition: 5.3 A function f: X → Y is said to be locally contra-wg-continuous provided that for every $x \in X$ and for every wg-open subset V of Y containing f(x), there exists a wg-closed subset U of X containing x such that $f(U) \subseteq V$.

Theorem: 5.4 If f: X → Y is locally contra-wg-continuous and Y is wg-T_2 space, then G(f) is locally contra-wg-closed in $X \times Y$.

Proof: Let $(x, y) \in (X \times Y) - G(f)$, then $y \neq f(x)$. Since Y is wg-T_2 space, there exist wg-open sets V and W in Y such that $f(x) \in V$ and $y \in W$ and $V \cap W = \Phi$. Since f is locally contra-wg-continuous, there exist an wg-closed set U in X containing x such that $f(U) \subseteq V$ which implies that $f(U) \cap W = \Phi$. Hence by the Lemma 4.2, G(f) is locally contra-wg-closed in $X \times Y$.

Theorem: 5.5 If f: X → Y is a injection with a locally contra-wg-closed graph, then X is wg-T_1 space.

Proof: Let x and y be two distinct points of X. Then $f(x) \neq f(y)$. Since f has a locally contra-wg-closed graph, there exist a wg-closed set D in X containing x and a wg-open set V in Y containing f(y) such that $f(D) \cap V = \Phi$. This menas that $y \notin D$ and therefore X is wg-T_1 space.

Definition: 5.6 A subset A of X is said to be WG-Closed relative to X, if every cover of A by wg-closed sets of X has a finite subcover.

Theorem: 5.7 If f: X → Y is locally contra-wg-closed graph, f(H) is wg-closed in Y for each subset H is WG-Closed relative to X.

Proof: Suppose that y is a point in Y \ f(H). We have $(x, y) \in (X \times Y) - G(f)$ for each $x \in H$. Since G(f) is locally contra-wg-closed, there exists a wg-closed subset D_x of X containing x and a wg-open set V_x of Y containing y such that $f(D_x) \cap V_x = \Phi$. The family $\{D_x / x \in H\}$ is a cover of H by wg-closed sets of X. Then, there exists a finite subset H_0 of H such that $H \subset \cup\{D_x / x \in H_0\}$. Set $V = \cap\{V_x / x \in H_0\}$. Now we have $(f(H) \cap V) \subset \cup_{x \in H0}(f(D_x) \cap V) \subset \cup_{x \in H0}(f(D_x) \cap V_x) = \Phi$. This shows that $y \notin Cl(f(H))$ and f(H) is wg-closed in Y.

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