Circular Choosability of Graphs

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Abstract- *This Paper attempts to study the concept of circular choosability of graph G (also called list chromatic number or choice number) It is denoted by* $\chi_l(G)$, *which is the minimum k such that G is t-choosable. It should be clear that* $\chi(G) \leq \chi_l(G) \leq \Delta(G) + 1$. *An upper bound of* $\chi_l(G)$ *in terms of* $\chi(G)$ *does not exist, since there are bipartite graph with arbitrarily large chromatic number. The chromatic number can be derived from the choosability of graphs.*

Keywords- Chromatic number, Circular Coloring, circular list assignment, k-degenerate graph.

I. INTRODUCTION

Suppose $G=(V,E)$ is a graph and $p \geq 2q$ are positive integers. A (p,q)- coloring of *G* is a mapping $f:V \to \{0,1,\cdots,p-1\}$ such that for any edge *uv* of *G*, $q \leq |f(u)-f(v)| \leq p-q$. Note that a $(p,1)$-coloring of a graph *G* is the same as a *p*-coloring of *G*. The *circular chromatic number* $\chi_c(G)$ of *G* is defined as $\chi_c(G)=\inf\{p/q : G \text{ admits a } (p,q)-\text{coloring}\}$.It is known that for any graph *G*, $\chi(G)-1<\chi_c(G)\leq\chi(G)$.So $\chi_c(G)$ is a refinement of $\chi(G)$ and $\chi(G)=\lceil\chi_c(G)\rceil$ is an approximation of $\chi_c(G)$.Let *C* be a set of integers (called colors). A list assignment *L* is a mapping which assigns to each vertex *v* of *G* a subset $L(v)$ of *C*. The set $L(v)$ is called the set of permissible colors for *v*. An *L*-coloring of *G* is a mapping $f:V\to C$ such that for each vertex *v*, $f(v)\in L(v)$ and for each edge uv, $f(u)\neq f(v)$. The list chromatic number or the choosability $\chi_l(G)$ of *G* is the least integer *k* such that for any list assignment *L* for which $|L(v)|=k$ for every vertex *v* of *G*, there is an *L*-coloring of *G*.

II. PRILIMINARIES

Definition: 2.1.1

Suppose G is a graph and $p\geq 2q$ are positive integers. A (p,q)-list assignment L is a mapping which assigns to each vertex v of G a subset $L(v)$ of $\{0,1,\cdots,p-1\}$. An $L(p,q)$-coloring of G is a (p,q)-coloring f of G such that for any vertex v, $f(v)\in L(v)$.

Definition: 2.1.2

A list size assignment is a mapping $l:V\to[0,p/q]$. Given a list size assignment l an $l-(p,q)$ list assignment is a (p,q) list assignment L such that for each vertex v, $|L(v)|\geq l(v)q$. A graph G is called $l(p,q)$- choosable if for any $l-(p,q)$ list assignment L, G has an $L-(p,q)$- coloring.

Definition: 2.1.3

A subset U of $S(r)$ is said to be assignable if it is the union of finitely many disjoint open arcs on $S(r)$. The length of an assignable set U, denoted by length (U), is the sum of the lengths of the open arcs of U.

If $G=(V,E)$ is a graph, then a function L that assigns to each vertex v of G an assignable subset $L(v)$ of $S(r)$ is called an circular list assignment (with respect to r).

If for each vertex v of G, $L(v)$ has length at least t, then L is called a t-circular list assignment (with respect to r).

A circular L-coloring of G is a mapping c from V to $S(r)$ such that $C(v) \in L(v)$ for each vertex v of G and for every pair (u,v) of adjacent vertices of G, $|C(u)-C(v)|_r \geq 1$.

2.2 BASIC PROPERTIES

Let $S(r)$ be a circle of circumference r, whose elements of $S(r)$ are identified with points in the interval $[0,r)$. For $a,b \in S(r)$, (a,b) denotes the open interval of $S(r)$ from a to b along the increasing direction.

To be precise, if $a \leq b$, then $(a,b)=\{x \in [0,r): a < x < b\}$ (So $(a,a)=\phi$).

If $a > b$, then $(a,b)=\{x \in [0,r): a<x<r\} \cup \{x \in [0,r): 0 \leq x < b\}$. The closed interval $[a,b]$ is defined similarly.

Definition: 2.2.1

A $t-(p,q)$- list assignment is a (p,q)- list assignment L such that for every vertex v, $|L(v)| \geq tq$. We say G is circular $t-(p,q)$- choosable if for any $t-(p,q)$- list assignment L, G has an $L-(p,q)$- coloring. We say G is circular t-choosable if G is circular $t-(p,q)$- choosable for any positive integers $p \geq 2q$.

The circular list chromatic number (or the circular choosability) of G is defined as
$\chi_{c,l}(G) = \inf\{t : G \text{ is circular } t\text{- choosable}\}$.

List Circular Chromatic Number of a Graph:

Lemma: 2.2.2

Suppose G is a graph and t is a positive real number. If for every t-circular list assignment L, G has a circular L-coloring, then G is circular t-choosable. Conversely, if G is circular t- choosable, then for every $\in > 0$ for any $(t+\in)$- circular list assignment L, G has a circular L-coloring.

Proof: Assume for every t-circular list assignment L, G has a circular L-coloring.

Let L' be an arbitrary $t-(p,q)$- list assignment.

Let L be the t- circular list assignment with respect to $r = p/q$ defined as,

$$L(v) = \bigcup_{i \in L'(v)} \left(\frac{i}{q}, \frac{i+1}{q} \right)$$

By assumption G has a circular L-coloring of f.

Let $f'(v) = [f(v)q]$, then f' is an $L'-(p,q)$- coloring of G.

Conversely, Assume that G is a circular t-choosable.

i.e.) for every positive integers $p \geq 2q$, for every $t-(p,q)$ - list assignment L of G, there is an $L-(p,q)$- coloring of G.

Let $\in > 0$ and let L' be a $(t+\in)$- circular list assignment with respect to r.

Let q be a fixed positive integer.

For a vertex v of G, Let $L(v) = \{i \in Z;\ i/q \in L(v)\}$.

Since $L'(v)$ has length at least $t+\in$, if q is sufficiently large, then $|L(v)| \geq tq$.

Let $p = [rq]$, then L is a $t-(p,q)$- list assignement. By assumption, there is an $L-(p,q)$- coloring f of G. It is straight forward to verify that $f'(v) = \frac{f(v)}{q}$ is a circular L'- coloring of G.

Hence the proof.

Lemma: 2.2.3

The following are equivalent formulations of cch, respectively cch_m.

(i) $\text{cch}(G) = \inf\{t \ge 1 : G \text{ is } t - \text{strict circular choosable}\}$

(ii) $\text{cch}_m(G) = \inf\{t \ge 1 : G \text{ is } (t,m) - \text{strict circular choosable}\}$

Proof: The proof of (i):

We shall only give the proof of (i). Because, the proof of (ii) is completely analogous.

Let $\tau(G)$ denote the,

$\inf\{t \ge 1 : G \text{ is } t - \text{strict circular choosable}\}$

Clearly, $\text{cch}(G) \le \tau(G)$

Since strict circular colorings are also circular colorings.

Now, let L be a list assignment that does not allow a strict circular coloring.

Let $0 < \in < 1$ be arbitrary.Let us set $r := r(L)$ $r' := (1-\in)r$ and define L' by setting $L'(v) = (1-\in)L(v) \subseteq S(r')$.

We claim that L' does not allow a (non-strict) circular coloring.

From this it will follow that, $\text{cch}(G) \ge t(L')$

$= (1-\in)t(L)$

and, since $L, \in$ are arbitrary, it also show that $\text{cch}(G) \ge \tau(G)$.

To prove the claim,

Suppose that $c' : V \to S((1-\in)r)$ is a (non-strict) circular coloring with $c'(v) \in L'(v)$

and Let $c : V \to S(r)$ be given by

$$c(v) := \frac{c'(v)}{(1-\varepsilon)}$$

If $uv \in E$ is an edge then

$$|c(u) - c(u)|_r = \min(|c(v) - c(u)| r - |c(v) - c(u)|)$$

$$= \frac{\min(|c'(v) - v'(u)| r' - |c'(v) - c'(u)|)}{(1-\varepsilon)}$$

$$= \frac{|c'(v) - c'(u)| r'}{(1-\varepsilon)}$$

$$\ge \frac{1}{(1-\varepsilon)}$$

Using that $|\lambda x - \lambda y| = \lambda |x - y|$ for all $x, y \in R$ and $\lambda \ge 0$.

Thus c is a strict circular coloring with $c(v) \in L(v)$ for all v, contradicting the choice of L.So the claim holds indeed.

Theorem: 2.2.5

The odd cycle C_{2k+1} has $\chi_{c,l}(C_{2k+1}) = 2 + \frac{1}{k}$.

Proof: Since $\chi_c(C_{2k+1}) = 2 + \frac{1}{k}$

By lemma,

"For any graph G, $\chi_c(G) \le \chi_{c,l}(G)$".

We have $\chi_{c,l}(C_{2k+1}) \ge 2 + \frac{1}{k}$.

Let L be a $t-(p,q)$ - list assignment, where $t \ge \frac{(2k+1)}{k}$.

We shall prove that, C_{2k+1} is $L-(p,q)$ - colorable.

Assume that the vertices of C_{2k+1} are $v_0, v_1, \cdots, v_{2k}$ and the edge are $v_i v_{i+1}$ for $i = 0, 1, \cdots, 2k$ (addition modulo of $2k+1$).

Let c_0 be any color in $L(v_0)$.

Let $L'(v_1) = \frac{L(v_1)}{[c_0 - q + 1, c_0 + q - 1]}$

i.e. delete from $L(v_1)$ all the colors that lie in the interval $[c_0 - q + 1, c_0 + q - 1]$ (Calculation modulo p and if $a > b$, then the interval $[a, b]$ denotes the set $\{i : a \le i \le p-1, \text{ (or) } 0 \le i \le b\}$).

As $[c_0-q+1,c_0+q-1]$ contains $2q-1$ elements and $L(v_1)$ contains at least $2q+\frac{q}{k}$ elements,

We conclude that $L'(v_1)=\frac{L(v_1)}{[c_0-q+1,c_0+q+1]}$ contains atleast $1+\left[\frac{q}{k}\right]$ elements.

Let $S=\left[\frac{q}{k}\right]$,Assume $\{c_{1,1},c_{1,2},\cdots,c_{1,S+1}\}\subseteq L'(v_1)$

Let $L'(v_2)=\frac{L(v_2)}{\bigcap_{i=1}^{S+1}[c_{1,i}-q+1,c_{1,i}+q-1]}$

i.e., delete from $L(v_2)$ all the colors that lie in the intersection $\bigcap_{i=1}^{S+1}[c_{1,i}-q+1,c_{1,i}+q-1]$.

We may assume that, $c_{1,1}<c_{1,2}<\cdots,c_{1,S+1}$

Then,

$$\bigcap_{i=1}^{S+1}[c_{1,i}-q+1,c_{1,i}+q-1]\subseteq[c_{1,1}+S+1-q,c_{1,1}+q-1]$$

this implies that, $\bigcap_{i=1}^{S+1}[c_{1,i}-q+1,c_{1,i}+q-1]$ contains atmost $2q-S-1$ elements.

Hence, $L'(v_2)$ contains at least $1+2S$ elements.

Continuing this, Let $L'(v_i)=\frac{L(v_i)}{\bigcap_{c\in L'(v_{i-1})}[c-q+1,c+q-1]}$

Then by induction, we can prove that

$$|L'(v_i)|\geq 1+is$$

In particular,

$$|L'(v_{2k})|\geq 1+2ks$$

$$\geq 1+2k\frac{q}{2k}$$

[here $S=\frac{q}{k}$]

$$=1+2q$$

Hence, $\frac{L'(v_{2k})}{[c_0-q+1,c_0+q-1]}\neq\phi$

Let c_{2k} be any color $c_{2k-1}\in L'(v_{2k-1})$ such that $c_{2k}\notin[c_{2k-1}-p+1,c_{2k-1}+p-1]$

Suppose $i\geq 2$ and we have chosen $c_i\in L'(v_i)$ then choose c_{i-1} to be any color in

$L'(v_{i-1})$ for which $c_i\in[c_{i-1}-p+1,c_{i+1}+p-1]$

.Then by definition of $L'(v_i)$ such a color exists.Then the color v_i with color $f(v_i)=c_i$ for $i=0,1,\cdots,2k$.

By the choice of these colors, f is a (p,q)- L-coloring C_{2k+1} .

Hence the theorem.

Lemma: 2.26

Suppose G is a graph and k is an integer. If $\chi_l(G)>k$ then $\chi_{c,l}(G)\geq k$.

Proof: Let L be a list assignment, which assigns to each vertex v of G a set $L(v)$ of at least k permissible colors.

Let $r\geq\max\{i+1:\ i\in L(v)\text{ for some }v\in V\}$.

Let L' be a circular list assignment with respect r of G defined as, $L'(v)=\bigcup_{i\in L(v)}(i,i+1)$

Then it is easy to see that if G is not L-colorable.

Then G is not circular L'-colorable.

Hence the proof.

2.3 CIRCULAR LIST COLORING OF k-DEGENERATE GRAPH:

If k-degenerate graphs G have $\chi_l(G)\leq k+1$.

We conclude that the difference $\chi_{c,l}(G)-\chi_l(G)$ can be arbitrary large

Then the ratio $\frac{\chi_{c,l}(G)}{\chi_l(G)}$ for the graph given in the proof of theorem 2.3.2 can be arbitrary close to 2. It is not clear whether the ratio $\frac{\chi_{c,l}(G)}{\chi_l(G)}$ is bounded.

Theorem: 2.3.1

Suppose G is a finite k-degenerate graph and L is a 2k-circular list assignment of G. Then there is a mapping f which assigns to each vertex v of G an open interval $f(v)\subseteq L(v)$ of positive length such that $v\sim v'$, then for any $x\in f(v)$ and $x'\in f(v')$, $|x-x'|_r\geq 1$.

Proof: We prove the result by induction on the number of vertices.

If $|V(G)|=1$ then this is certainly true.

Assume $|V(G)|=n$ and the result holds for any graph G' on $n-1$ vertices.

Let v be a vertex of G with $d_G(v)=K'$, $d_G(v)\leq k'$, $k'\leq k$

Let $G'=G-v$

Then G' is k-degenerate and hence the required mapping f for G' exists by induction hypothesis.

Let $v_1,v_2,\cdots,v_k$ be the neighbours of v.

Assume $f(v_i)=(a_i,b_i)$,for $i=1,2,\cdots,k'$.

Without loss of generality, we may assume that (a_i,b_i) has length less than 2.

(If (a_i,b_i) has length greater than 2, we can replace it by a sub-interval of (a_i,b_i)).

For any point $x\notin[b_i-1,a_i+1]$ (here the calculations are modulo r) there is a point $y\in(a_i,b_i)$ such that, $|x-y|_r>1$

The interval $[b_i-1,a_i+1]$ has length less than 2.

So the union $\bigcup_{i=1}^{k'}[b_i-1,a_i+1]$ has length less than $2k'\leq 2k$.

As $L(v)$ has length at least $2k$ it follows that there is a point $x\in\frac{L(v)}{\bigcup_{i=1}^{k'}[b_i-1,a_i+1]}$.

Now for each $i\in\{1,2,\cdots,k'\}$ there is a point $y_i\in(a_i,b_i)$ such that $|x-y_i|_r>1$.

Note that the set $\frac{L(v)}{\bigcup_{i=1}^{k'}[b_i-1,a_i+1]}$ is open.

So for each $i\in\{1,2,\cdots,k'\}$ there is an open interval $(c_i,d_i)\subseteq\frac{L(v)}{\bigcup_{i=1}^{k'}[b_i-1,a_i+1]}$ containing x and an open interval $(a_i',b_i')\subseteq(a_i,b_i)$ containing y_i such that for any $x'\in(c_i,d_i)$ and for any

$$y_i'\in(a_i',b_i'),|x'-y_i'|_r>1$$

We modify f and then extend it to $V(G)$ by as follows.

$$f(v_i)=(a_i',b_i') \text{ and } f(v)=\bigcap_{i=1}^{k'}(c_i,d_i)$$

Then it follows from the definition that the resulting mapping is a required mapping for G.

Hence the proof.

Theorem: 2.3.2

For any positive integer k, for any $\in > 0$, there is a k-degenerate graph G and a $(2k-\in)$ circular list assignment L for which there is no circular L-coloring of G.

Proof: Let n be an integer that $n \in > 2k^2$.

Let $G = K_{k,n^k}$ be the complete bipartite graph with partite sets, $A = \{u_1, u_2, \cdots, u_k\}$ and $B = \{v_{j_i, j_2, j_k};\ 1 \le j_i \le n\}$

It is obvious that G is k-degenerate.

Let $r = 2k(k+1)$. For $i = 1, 2, \cdots, k$.

Let $a_i = (i-1)(2k+2)$ and $\delta = \frac{2k}{n} < \frac{\in}{k}$

Define a circular list assignment L of G as follows.

For $i = 1, 2, \cdots, k$ Let $L(u_i) = (a_i, a_i + 2k)$

Let $L(v_{j_1, j_2, \cdots, j_k}) = \bigcup_{i=1}^{k} A_{i, j_i}$

[Where $A_{i,j_i} = (a_i + j_i\delta - 1, a_i + (j_i - 1)\delta + 1)$].

Note that A_{i,j_i} is an interval of length $2-\delta$ and that $A_{i,j_i} \cap A_{i',j_i'} \ne \phi$ if $i \ne i'$.

So $L(V_{j_1, j_2, \cdots, j_k})$ has length $(2-\delta)k > 2k - \in$.

So L is a $(2k-\in)$- circular list assignment of G.

We shall prove that G is not circular L-colorable.

Assume to the contrary that c is a circular L-coloring of G.

For $i \in \{1, 2, \cdots, k\}$. Let $1 \le j_i \le n$ be an integer such that $c(u_i) \in [a_i + (j_i - 1)\delta, a_i + j_i\delta]$.As $c(u_i) \in L(u_i) = (a_i, a_i + 2k)$ such an integer j_i exists.

For any $i \in \{1, 2, \cdots, k\}$ as $v_{j_1, j_2, \cdots, j_k}$ is adjacent to u_i.
We conclude that,
$c(v_{j_1, j_2, \cdots, j_k}) \notin (a_i + j_i\delta - 1, a_i + (j_i - 1)\delta + 1) = A_{i,j_i}$

for any $i \in \{1, 2, \cdots, k\}$.This is a contradiction, as $L(v_{j_1, j_2, \cdots, j_k}) = \bigcup_{i=1}^{k} A_{i,j_i}$

Hence the proof.

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