Fuzzy Dimensions of Fuzzy Vector Space

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Abstract- *This paper attempts to study how the vector spaces can be used in Fuzzy and the relationship between fuzzy vectors and fuzzy bases, and also how the fuzzy dimensions can be used to derive the fuzzy vector. We prove that all finite dimensional vector spaces have fuzzy bases. Also it is proved that the two fuzzy vector spaces have the same fuzzy dimensions, and the summation of two fuzzy vector spaces is equal to the summation of fuzzy dimensions.*

Keywords- Vector space, bases, fuzzy linear, fuzzy dimensions.

I. INTRODUCTION

In set theory the sets are considered as abstract sets which are defined as collection of objects having some very general property. But in fact, most of the classes of objects uncounted in the real physics work are of fuzzy type and not sharply defined. Thus they don't have precisely defined criteria of membership. In such a class an object need not necessarily either belong to or not belong to a class: there may be intermediate grades of membership. This is the concepts of fuzzy sets, which is a "class" with a continuum of grades of membership.

The notation of fuzziness is introduced in group theory, ring theory vector spaces etc and thus fuzzy groups, fuzzy rings, fuzzy vector spaces are generated.

In this project some algebraic properties of fuzzy vector spaces are discusses.

II. PRELIMINARIES

In this chapter some definitions and results which are needed to develop the project are stated.

Definition 2.1:

A non empty set V is said to be a vector space over a field F, if V is an abelian group under an operation called addition which we denote by + and for every $\alpha \in F$ and $v \in V$ there is defined an element $v\alpha$ in V subject to the following conditions.

(i) $\alpha(u + v) = \alpha u + \alpha v \ \forall u, v \in V$ and $\alpha \in F$
(ii) $(\alpha + \beta)u = (\alpha u + \beta u) \ \forall u \in V, \ \alpha, \beta \in F$
(iii) $\alpha(\beta u) = (\alpha\beta)u \ \forall u$ and $\alpha, \beta \in F$
(iv) $I u = u, \ \forall u \in V$.

Definition 2.2:

If V is a vector space over a field F, a non-empty subset W of V is called a sub space of V if W itself form a vector space over F with respect to the addition and scalar multiplication already defined in V.

Definition 2.3:

Let V be a vector space over a field F. let $v_1, v_2 \ldots\ldots v_n \in V$. Then any element of the form $\alpha_1 v_1 + \alpha_2 v_2 + \ldots.. + \alpha_n V_n$ where $\alpha_i \in F$ is called a linear combination of the vectors $v_1, v_2 \ldots v_n$.

Definition 2.4:

Let S be a non-empty subset of a vector V. Then the set of all linear combination of finite sets of element of S is called the linear span of S and is denoted by L(S).

Definition 2.5:

A set of vectors $v_1, v_2 \ldots v_n$ belonging to a vector space V is said to be linearly dependent over F if there exists scalars $\alpha_1, \alpha_2 \ldots \alpha_n$ in F not all zero such that, $\alpha_1 v_1 + \ldots\ldots.. + \alpha_n v_n \neq 0$.

If the vectors $v_1, v_2 \ldots v_n$ are not linearly dependent over F, they are said to be linearly independent over F.

(i.e) $\alpha_1 v_1 + \ldots\ldots.. + \alpha_n v_n = 0$ if and only if α_i' s are zero.
(i.e) $\alpha_1 = \alpha_2 = \ldots.. = \alpha_n = 0$

Definition 2.6:

The set $\{ v_1, v_2 \ldots v_n \}$ of element of a vector space V over a field F is said to be a basis of V, if the vectors $v_1, v_2, v_2 \ldots\ldots v_n$ are linearly independent and if V is generated by $v_1, v_2 \ldots v_n$

Definition 2.7:

The number of elements in the basis of a vector space is called the dimension of the vector space. If that number is finite then we say that the vector space is finite dimensional otherwise it is of infinite dimensions.

Theorem 2.8:

Let V be a finite dimensional vector space over a field F. let A and B two subspace of V then
dim (A+B) = dim A + dim B-dim (A∩B).

Definition 2.9:

A fuzzy set in X is a function with domain X and values in I. That is, an element of I^x.

Let A Є I^x the subset of X in which a assumes non-zero values is known as support of A.For every x Є X , A(x) is called the grade of membership of x in A.X is called the carrier of the fuzzy set A.

If A takes values 0 and 1 then A is called a crisp set in X.

Note: 2.10

A member A of I^x is contained in a member B of I^x denote A≤B if and only if A(x) ≤ B(x), for every x Є X.

Definition 2.11:

Let A ,B Є I^x. We define the following fuzzy sets.

Union : AUB Є I^x by (AUB)(x)=max{A(x),B(x)}, for every x Є X

Intersection: A∩B Є I^x by
(A∩B)(x) =min{A(x),B(x)},for every x Є X

Complement :A^c Є I^x by
A^c (x) = 1-A(x), for every x Є X.

Definition 2.12:

Let f: X Є Y, A Є I^x and B Є I^y ,then f(A) is a fuzzy set in Y,

Defined by

$$f(A)(y) = \begin{cases} \sup\{A(x);\ x \in f^{-1}(y)\}, & \text{if } f^{-1}(y) \neq 0 \\ 0, & \text{if } f^{-1}(y) \neq 0 \end{cases}$$

And $f^{-1}(B)$ is a fuzzy set in X, defined by

$f^{-1}(B)(x) = B(f(x))$, x Є X.
Then,
1. $f^{-1}(B^c) = (f^{-1}(B))^c$, for any fuzzy set B in Y
2.$f(f^{-1}(B)) \leq$ B, for any fuzzy set B in Y
3.$A \leq f^{-1}(f(A))$, for any fuzzy set A in X.

Definition 2.13:

The product $f_1*f_2 : X_1*X_2 \rightarrow Y_1*Y_2$ of mappings,
$f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ is defined by
$(f_1*f_2)(x_1,x_2) = (f_1(x_1),f_2(x_2))$

Definition 2.14:

For a mapping f : X→ Y, the graph g :X →X*Y of f is defined by g(x)=(x, f(x)), for each x Є X

Definition 2.15:

Let A Є I^x and B Є I^y , then by A*B. We denote the fuzzy set in X*Y for which (A*B) (x, y) = min (A(x),B(y)),for every (x, y) Є X*Y

Definition 2.16:

The power set p (I) of the given set I is the set of all subsets of I.

Definition 2.17:

A non-empty set p, together with a binary relation R is said to form a partially ordered set or poset if the following conditions hold.

P1: Reflexivity: aRa of all a Є p.
P2: Anti-Symmetry : If aRb, bRa, then a=b where a, b Є p.
P3: Transitivity: If aRb and bRc then aRc where a,b,c Є p. For conveience we use the symbol ≤ in the place of R.

III. FUZZY VECTOR SPACES

Definition 3.1:

Let E be vector space. Then the fuzzy vector space is pair Ẽ = (E, μ) where μ : E → [0,1] with the properties that for all a, b Є R and x, y Є E we have
$\mu(ax + by) \geq \mu(x) \cap \mu(y)$

Definition 3.2:

If Ẽ = (E, μ) is a fuzzy vector space, then we have the following sets defined as
$T_{\mu}^{\circ} = \mu^{-1}(\alpha)$, $H_{\mu}^{\circ} = \mu^{-1}([\alpha,1])$ and $E_{\mu}^{\circ} = \mu^{-1}([\alpha,1])$ for all a Є R\{0}, μ(ax)= μ(x)

Properties 3.3:

If $\tilde{E} = (E, \mu)$ is fuzzy vector space then

(i) $H_{\mu}^{\circ} < H_{\mu}^{\alpha} < E$,

(ii) If $u,v \in E$ and $\mu(u) > \mu(v)$ then $\mu(u + v) = \mu(v)$

Proof :

Let us prove (i) From the definition we have

$H_{\mu}^{\alpha} = \mu^{-1}([\alpha,1])$

$\Rightarrow \mu(H_{\mu}^{\alpha}) = ([\alpha,1)] \longrightarrow (1)$

and $E_{\mu}^{\alpha} = \mu^{-1}([\alpha,1])$

$\Rightarrow \mu(E_{\mu}^{\alpha}) = ([\alpha,1]) \longrightarrow (2)$

from (1) and (2)

$(\alpha, 1) \subset [\alpha,1]$

(i.e) $\mu^{-1}((\alpha,1]) \subset \mu^{-1}([\alpha, 1])$

(i.e) $H_{\mu}^{\alpha} \subset E_{\mu}^{\alpha}$

$H_{\mu}^{\alpha} < E_{\mu}^{\alpha}$

$\Rightarrow$ Since H_{μ}^{α}, E_{μ}^{α} are subspaces of E, we have $H_{\mu}^{\alpha}, < E_{\mu}^{\alpha} < E$.

Proof (ii):

If $u, v \in E$ and $\mu(u) > \mu(v)$ then $\mu(u + v) = \mu(v)$ by the definition of fuzzy vector space we have

$\mu(u + v) \geq \mu(u) \cap \mu(v)$

$\geq \min(\mu(u), \mu(v))$

$\geq \mu(u)$

$\mu(u + v) \geq \mu(v) \longrightarrow (1)$

Also

$\mu(v) = \mu(u + v - u) \geq \mu(u + v) \cap \mu(u)$

$\geq \min\{\mu(u + v), \mu(u)\}$

$\geq \mu(u + v) \longrightarrow (2)$

From (1) and (2) we have, $\mu(u + v) = \mu(v)$

Proposition 3.4:

If $\tilde{E} = (E, \mu)$ is a fuzzy vector space and if $v,w \in E$ with $\mu(v) \neq \mu(w)$ then

$\mu(v + w) = \mu(v) \cap \mu(w)$.

Proof :

Since $\mu(v) \neq \mu(w)$

Then either $\mu(v) > \mu(w)$ or $\mu(v) < \mu(w)$

Case (1):

If $\mu(v) > \mu(w)$ then

$\mu(v + w) = \mu(v) \longrightarrow (1)$

(by proposition 3.3)

Case (2):

If $\mu(w) > \mu(v)$ then

$\mu(v + w) = \mu(w) \longrightarrow (2)$

from (1) and (2) we have

$\mu(v + w) = (\min \mu(v), \mu(w))$

$= \mu(v) \cap \mu(w)$

Proposition 3.5:

If $\tilde{E} = (E, \mu)$ is a fuzzy vector space then $\mu(0) = \sup\limits_{x \varepsilon E} \mu(x) = \sup[\mu(E)]$

Proof :

Let $x \in E$, $\mu(0) = \mu(0x)$

if $a \varepsilon R$, then $\mu(ax) \geq \mu(x)$

$\mu(0x) \geq \mu(x)$

$\geq \sup\limits_{x\varepsilon E} \mu(x)$

$\mu(0) \geq \sup\limits_{x\varepsilon E} \mu(x)$

$\Rightarrow \mu(0) \geq \sup[\mu(E)] \longrightarrow (1)$

But $\sup \mu(x) \geq \mu(x)$ for all $x,\mu \in E$, for $x = 0$

$\sup \mu(E) \geq \mu(0) \longrightarrow (2)$

from (1) and (2) we have

$\mu(0) = \sup\limits_{x\varepsilon E} \mu(E) = \sup\limits_{x\varepsilon E} \mu(x)$

Definition 3.6: (3)

Let S: P(R+ u{0} → R_+ u {0} {∞} where p(x) denotes the power set of x such that $S(A) = \sum\limits_{A \varepsilon A} a$

In case $A \cap R_+$ is uncountable we must have

$S(A)=\alpha$

IV.FUZZY LINEAR INDEPENDENCE

Definition 4.1:

Let $\tilde{E} = (E,\mu)$ be a fuzzy vector space. We say that a finite set of vectors $\{X_i\}_{i=1}^{n}$ is fuzzy linear independent in $\tilde{E}$ if and only if $\{X_i\}_{i=1}^{n}$ is linearly independent in E and for all $\{a_i\}_{i=1}^{n}$ is R.

$$\mu\{\sum_{i=1}^{N} a_i x_i\} = \bigcap_{i=1}^{n} \mu(a_i x_i)$$

Example 4.2:

Consider $\tilde{E} = (R^2, \mu)$ where $\mu[0,0] = 1$, $\mu[0,R\backslash\{0\}]=1/2$ and $\mu(R^2|\ (0,R]) = 1/4$.

we shall show that the vectors X = (1,0) and y=(-1,1) are linearly independent in $\tilde{E}$. This example also illustrates a situation where

$\mu(x) = \mu(y)$ and $\mu(x + y) > \mu(x)$

For if x = (1,0) and y = (-1,1)

Consider $\alpha x + \beta y = 0$

Then $\alpha (1,0) + \beta(-1,1) = (0,0)$

(i.e) $(\alpha,0) + (-\beta, \beta) = (0,0)$

$(\alpha-\beta, \beta) = (0,0)$

$\Rightarrow \alpha-\beta = 0, \beta=0$

$\alpha =\beta, \beta=0$

$\Rightarrow \alpha = 0, \beta=0$

x and y are linearly independent

$\mu(x) = \mu(1,0) = 1/4$

$\mu(y) = \mu(-1,1) = 1/4$

$\mu(x) = \mu(y) \longrightarrow (1)$

$x + y = (1,0) + (-1,1) = (0,1)$

$\mu (x + y) = 1/2 \longrightarrow (2)$

from (1) and (2)

$\mu (x + y) > \mu(x)$

(i,e) $\mu (x + y) \neq \mu (x) \cap \mu(y)$

The vectors are not fuzzy linearly independent.

Proposition 4.3:

Let $\tilde{E} = (E, \mu)$ be a fuzzy vector space. Then any set of vector $\{X_i\}_{i=1}^{N} \subset E\backslash\{0\}$ which has distinct μ value linearly and fuzzy linearly independent.

Proof:

We prove the proposition by induction on N.

If N=1 we have only one vector and clearly the statement is true. Suppose that the proposition is true for N. then we have to prove result for (N+1).

Let $\{X_i\}_{i=1}^{N+1}$ be a set of vectors in $E\backslash\{0\}$ with distinct μ values. By inductive hypothesis we have $\{X_i\}_{i=1}^{N}$ is fuzzy linearly independent. Suppose that $\{X_i\}_{i=1}^{N+1}$ is not linearly independent.

Then $X_{N+1} = \sum a_i x_i$, where $S \subset \{1,2,\ldots N\}$, $S\neq O$ and for all $I \in S$, $a_i \neq 0$

$$\mu(X_{N+1}) = \mu(\sum a_i x_i) = \bigcap_{i \in s} \mu(a_i x_i) = \bigcap_{i \in s} \mu(x_i)$$

And hence $\mu(X_{N+1}) \in \{\mu (X_i)\}_{i=1}^{N}$

This contradicts the fact that $\{X_i\}_{i=1}^{N}$ has distinct μ values. This contradiction arises because we have assumed that $\{X_i\}_{i=1}^{N+1}$ is not linearly linearly independent.

Therefore $\{X_i\}_{i=1}^{N+1}$ is linearly independent.

Now we have to show that $\{X_i\}_{i=1}^{N+1}$ is fuzzy linearly independent. Since, the μ values are different and $\mu(x)$, for all $a \neq 0$.

$$\mu\left(\sum_{i=1}^{N} a_i x_i\right) = \mu\left(\sum_{i=1}^{N} x_i\right) = \bigcap_{i=1} \mu(x_i)$$

$$\mu\left(\sum_{i=1}^{N+1} a_i x_i\right) = \mu\left(\sum_{i=1}^{N+1} x_i\right) = \mu\left(\sum_{i=1}^{N} x_i + X\right)$$

$$= \mu\left(\sum_{i=1}^{N} x_i\right) \cap \mu (X_{N+1})$$

$$= \bigcap_{i=1}^{N} \mu(x_i) \cap \mu (X_{N+1})$$

$$= \bigcap_{i=1}^{N+1} \mu(x_i)$$

$$= \bigcap_{i=1}^{N+1} \mu(a_i x_i)$$

$$\mu\left(\sum_{i=1}^{N+1} a_i x_i\right) = \bigcap_{i=1}^{N+1} \mu(a_i x_i)$$

Remarks 4.4:

If $\tilde{E} = (E, \mu)$ is fuzzy vector space such that dim E = n, then $|\mu(E)| \leq n+1$,Where $|\mu(E)|$ represents the cardinality of $\mu(E)$.

V. FUZZY BASES

In this chapter, we shall present a new definition of fuzzy bases for fuzzy vector spaces.

Let $\tilde{E} = (E,\mu)$ be a fuzzy vector space and dim E= n.

Definition 5.1:

Let N denote the set of all natural numbers and let L be a complete lattice. An L- fuzzy natural integer is an antitone mapping λ: N → L satisfying

$$\lambda(0) = T_L, \wedge \ \lambda(n) = \perp_L, \ n\in N$$

Where $\top_L$ and $\perp_L$ are the largest element and the smallest element in L, respectively. The set of all L-fuzzy natural integers is denoted by N(L).

Definition 5.2:

Let A be a fuzzy set, and define a map [A] : N →[0,1] ∋∀n ∈N ,
$|A|(n) = \vee \{a \in (0,1] : |A_{[a]}| \geq n\}$. Then $|A| \in N([0,1])$, which is called the cardinality of A.

Definition 5.3:

For any ⋌,μ∈N ([0,1]), the addition
⋌+ μ of ⋌ and μ is defined as follows:

$$(⋌ + \mu)(n) = \vee_{k+l=n}(⋌(k) \wedge \mu(l))$$,for any n ∈ N

Lemma 5.4 :

If E =(E,μ) is a fuzzy vector space, then there exists a finite sequence $1=\alpha_0 \geq \alpha_1 > \alpha_2 > \ldots > \alpha_r \geq 0$ such that
(i) If a,b ∈ $(\alpha_{i+1},\alpha_i]$, then $\mu_{[a]} = \mu_{[b]}$
(ii) If a ∈ $(\alpha_{i+1},\alpha_i]$, and b ∈ $(\alpha_i,\alpha_{i\,i-1}]$, then $\mu_{[a]} \supsetneq \mu_{[b]}$

For a fuzzy vector space Ẽ = (E,μ), by this Lemma **,** we can obtain a family of vector space as follows: $\{0\} \subseteq \mu_{[\alpha 1]} \subsetneq \mu_{[\alpha 2]} \subsetneq \ldots \subsetneq \mu_{[\alpha r]} \subseteq E$

The family of irreducible level subspaces of E = (E,μ). Suppose that $\mu_{[\alpha 1]} \neq \{0\}$, otherwise we can choose $\mu_{[\alpha 2]}$ .we can obtain a basis $B_{\alpha r}$ of $\mu_{[\alpha r]}$ by extending $B_{\alpha r-1}$. Thus we obtain a sequence.

$$B_{\alpha 1} \subsetneq B_{\alpha 2} \subsetneq B_{\alpha 3} \subsetneq \ldots \subsetneq B_{\alpha r}, \longrightarrow (1)$$

Where $B_{\alpha i}$ is a basis of $\mu_{[\alpha 1]}(1\leq i \leq r)$. therefore, we can define a fuzzy subset β of E as follows.
$\beta(x) = \vee\{\alpha_i : x \in B\,\alpha_i\}$, Then β is called a fuzzy basis of Ẽ corresponding to (1)
The proof of the following theorem is trivial.

Theorem 5.5:

Let β be a fuzzy basis of Ẽ = (E,μ) obtained by the above equation (1) . Then the following statements hold:

(i) If a,b ∈ $(\alpha_{i+1},\alpha_i]$, then $\beta_{[a]} = \beta_{[b]} = B_{\alpha i}$
(ii) If a ∈ $(\alpha_{i+1},\alpha_i]$, and b ∈ $(\alpha_i,\alpha_{i\,i-1}]$, then $\beta_{[a]} \supsetneq \beta_{[b]}$
(iii) If a,b ∈ $(\alpha_{i+1},\alpha_i]$, then $\beta_{(a)} = \beta_{(b)} = B_{\alpha+I}$
(iv) If a ∈ $(\alpha_{i+1},\alpha_i]$, and b ∈ $(\alpha_i,\alpha_{i\,i-1}]$, then $\beta_{[a]} \supsetneq \beta_{[b]}$

Corollary 5.6:

Let β be a fuzzy basis of Ẽ = (E,μ) obtained by the above equation (1) .Then the following statements hold:
(i) a ∈ (0,1], $\beta_{[a]}$ is a basis of $\mu_{[a]}$
(ii) a ∈ [0,1), $\beta_{(a)}$ is a basis of $\mu_{(a)}$

From the above corollary, we can easily obtain the following.

Corollary 5.7:

Let Ẽ = (E,μ) be a fuzzy vector space and let β_1 and β_2 be two fuzzy bases of Ẽ. then the following statements hold.
(i) For any a ∈ (0,1], $|(\beta_1)_{[a]}| = |(\beta_2)_{[a]}|$
(ii)For any a ∈ [0,1), $|(\beta_1)_{[a]}| = |(\beta_2)_{[a]}|$
(iii) $|\beta_1| = |\beta_2|$

Definition 5.8:

A fuzzy basis for a fuzzy vector space Ẽ =(E, μ) is a fuzzy linearly Independent basis for E.

Note 5.9:

The following theorem shows how we can construct a very wide class of fuzzy vector spaces with a fuzzy basis.

Theorem 5.10:

Given a vector space E with basis B={V, α }, α Є A, constant μ_0 Є (0,1] and any set of constants $\{\mu\alpha\}_{\alpha \in A}$ C (0,1] such that
$\mu_0 \geq \mu\alpha$ for all α Є A . Let us construct a function μ : E→[0,1] in the following way. Any Z ≠ 0,
Z Є E can be uniquely written as

$$Z = \sum_{i=1}^{N} a_i V \alpha_i \text{ with } a_i \neq 0$$

Define: $\mu(Z) = \bigcap_{i=1}^{N} \mu(V_{\alpha i}) = \bigcap_{i=1}^{N} \mu\alpha_i$ and $\mu(0)= \mu_0$

Cleary is defined for all Z Є E and is well defined we claim that Ẽ =(E, μ) is a fuzzy vector space with fuzzy basis B.

Proof:

Let X,Y Є E/{0}. Then X and Y can be uniquely in the following way

$$X= \sum_{i \in CUDx} X_i V_{\alpha i} \quad , \quad Y= \sum_{i \in CUDy} Y_i V_{\alpha i}$$

Such that C∩ D_x =φ, C∩ D_y =φ, Dx∩ D_y =φ, CU D_x and CU D_y are finite and non - empty and for all i Є CUD_x , X_i Є R\{0} and for all
i Є CUD_y,Y_i Є R\{0}.

Case (i) :

Suppose a,b ≠ 0 and a , b Є R, then ax + by ≠ 0

Let Z={i Є C\ $ax_i + by_i$ = 0} and N = C\Z

N = {i Є C\ $ax_i + by_i \neq 0$}

Let us suppose that C, D_x, D_y, Z and N are all non empty we shall prove the theorem for these sets. In case some of these sets are empty the proof of theorem is almost identical.

μ(ax+by)

$$= \mu\Big(\sum_{i \in C} (ax_i + by_i) V_{\alpha i} + \sum_{i \in Dx} (axi)V_{\alpha i} + \sum_{i \in Dy} (by_i)V_{\alpha i}\Big)$$

$$= \mu\Big(\sum_{i \in N} (ax_i + by_i) V_{\alpha i} + \sum_{i \in Dx} (axi)V_{\alpha i} + \sum_{i \in Dy} (by_i)V_{\alpha i}\Big)$$

all Co-efficient in the above linear combination are non-*zero* and thus by definition of μ we have,

μ (ax + by)

$$= \Big(\bigcap_{i \in N} \mu (V_{\alpha i})\Big) \cap \Big(\bigcap_{i \in Dx} \mu (V_{\alpha i})\Big) \cap \Big(\bigcap_{i \in Dy} \mu (V_{\alpha i})\Big)$$

$$= \Big(\bigcap_{\mathbf{i \in NUDxUDy}} \mu_{\alpha i}\Big) \cap \Big(\bigcap_{\mathbf{i \in N}} \mu_{\alpha i}\Big) \cap \Big(\bigcap_{\mathbf{i \in Dx}} \mu_{\alpha i}\Big)$$

$$= \bigcap_{\mathbf{i \in Dy}} \mu_{\alpha i}$$

$$\geq \bigcap_{i \in CUDxUDy} \mu_{\alpha i}$$

$$= \Big(\bigcap_{i \in CUDx} \mu_{\alpha i} \Big) \cap \Big(\bigcap_{i \in CUDy} \mu_{\alpha i}\Big)$$

$$= \mu(x) \cap \mu(y)$$

Therefore if a,b = 0 and ax+by=0, then
μ (ax+by) ≥ μ(x) ∩ μ(y).

case (ii):

when (ax+by)=0

since μ(0) = μ_0 ≥ sup μ(B) we must have

μ(ax+by) = μ(0) ≥ μ(x) ∩ μ(y)

case (iii):

When a or b is zero, without loss of generality we may say a=0

Then μ (0x+by) = μ (by) ≥ μ (x) ∩ μ(by)

≥ μ(x)∩ μ(y)

Therefore Ẽ=(E, μ) is a fuzzy vector space with fuzzy basics B

Hence the proof.

VI. FUZZY DIMENSION

In this section, we redefine the fuzzy dimension of fuzzy vector spaces.

The cardinality of a crisp set. A can be regarded as an increasing set of integers{0,1… ,n}. Such a set is also mathematically equivalent to the integer n. for a crisp vector space, its dimension was defined by the cardinality of its bases. We can define analogously the fuzzy dimension of fuzzy vector spaces as follows.

Definition 6.1:

Let Ẽ = (E, μ) be a fuzzy vector space with a fuzzy basis β. Define dim (Ẽ) = │β│. Then dim (Ẽ) is called the fuzzy dimension of
Ẽ = (E,μ).

Theorem 6.2:

Let Ẽ = (E, μ) be a fuzzy vector space with a fuzzy basis β. Then
dim (Ẽ) (n) = ∨ {α ∈[0,1) :
$│\beta_{(a)}│ \geq n$ }= ∨{α ∈[0,1): dim ($\tilde{E}_{(a)}$) ≥ n }

Proof:

we know that, dim ($\tilde{E}_{(a)}$) = $│\beta_{(a)}│$. For any n ∈ N ,
let ⋏= ∨ {α ∈[0,1): dim ($\tilde{E}_{(a)}$) ≥ n }Obviously, we have
⋏ ≤ dim (Ẽ) (n)= ∨ {α ∈(0,1] , $│\beta_{[a]}│$ ≥ n }.
In order to show that ⋏ ≥ dim (Ẽ)(n), suppose that dim (Ẽ)(n) ≠ 0 and dim (Ẽ) (n) > b. Then there exists a > b such that $│\beta_{[a]}│$ ≥ n. in this case, n ≤ $│\beta_{[a]}│$ ≤ $│\beta_{[b]}│$ ≤ $│\beta_{[b]}│$. This implies ⋏= ∨ {α ∈[0,1): dim $(\tilde{E})_{(a)}$) ≥ n } ≥b.Thus we have,⋏= ∨ {b: 0 ≤ b < dim (Ẽ) (n)}
= dim (Ẽ) (n).
This completes the proof.

Theorem 6.3:

Let Ẽ = (E,μ) be a fuzzy vector space. Then

(i) For any a ∈ (0,1], $(\dim(\tilde{E}))_{[a]} = \dim (\tilde{E}_{[\alpha]})$

(ii) For any a ∈ [0,1), $(\dim(\tilde{E}))_{[a]} = \dim (\tilde{E}_{[\alpha]})$

Proof :

Let $\{0\} \subseteq \mu_{[\alpha 1]} \subsetneq \mu_{[\alpha 2]} \subsetneq \ldots \subsetneq \mu_{[\alpha r]}$ E

Be the family of irreducible level subspaces of

$\tilde{E} = (E,\mu)$.

(i)We know from definition of dim $(\tilde{E})$ that dim $(\tilde{E}_{[\alpha]}) \leq$(dim $(\tilde{E}))_{[\alpha]}$. Now we need to show that (dim $(\tilde{E}))_{[\alpha]} \leq$ dim $(\tilde{E}_{[\alpha]})$. From the definition of fuzzy dimension , we have $n \leq (\dim(\tilde{E}))_{[a]}$

$\Longrightarrow (\dim(\tilde{E})(n) \geq \alpha$

$\Longrightarrow \bigvee \{\alpha_i : \dim(\tilde{E})_{[a]}) \geq n\} \geq \alpha$

$\Longrightarrow E_{\alpha i} \geq a$ such that $n \leq \dim(\tilde{E}_{[\alpha i]})$

$\Longrightarrow n \leq \dim(\tilde{E}_{[\alpha i]}) \leq \dim(\tilde{E}_{[\alpha]})$

Therefore, $(\dim(\tilde{E}))_{[\alpha]} = \dim(\tilde{E}_{[\alpha]})$ for any $a \in (0,1]$.

(ii)In order to prove $(\dim(\tilde{E}))_{(a)} \leq \dim(\tilde{E}_{(\alpha)})$, we suppose that $n = (\dim(\tilde{E}))_{(\alpha)}$, Then $(\dim(\tilde{E})(n)\ \alpha$,

i.e., $\bigvee \{b \in (0,1] : \dim(\tilde{E}_{[b]}) \geq n\} > \alpha$.

Hence, there exists $b \in (0,1]$ such that $a < b$ and $n \leq \dim(\tilde{E}_{[b]})$. Since $\tilde{E}_{[b]} \subseteq \tilde{E}_{(a)}$, thus $m \leq \dim(\tilde{E}_{[a]})$ therefore, dim $(\dim(\tilde{E}))_{(a)} \leq \dim(\tilde{E}_{(\alpha)})$

In order to show dim $(\dim(\tilde{E}))_{(a)} \leq \dim(\tilde{E}))_{(\alpha)}$, take $_{\alpha i} > {}_{\alpha}$ such that $\tilde{E}_{(\alpha)} = \mu_{[\alpha i]}$. Then it is easy to see that $\dim(\tilde{E}_{(\alpha)}) = \dim(\mu[_{\alpha i}])$

$\leq (\dim(\tilde{E}))_{(\alpha i)} \leq (\dim(\tilde{E}))_{(a)}$

Theorem 6.4 :

Let $\tilde{E}_1 = (E,\mu_1)$ and $\tilde{E}_2 = (E, \mu_2)$ be two fuzzy vector spaces, then it holds $(\dim(\tilde{E}_1+\tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2))_{(a)} = \dim(\tilde{E}_1)+\dim(\tilde{E}_2)$.

Proof :

$(\dim(\tilde{E}_1+\tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2)_{(a)}$

$= (\dim(\tilde{E}_1+\tilde{E}_2))_{(a)} + (\dim(\tilde{E}_1 \cap \tilde{E}_2))_{(a)}$ $= \dim((\tilde{E}_1+\tilde{E}_2)_{(a)}) + \dim((\tilde{E}_1 \cap \tilde{E}_2))_{(a)}$ $= \dim((\tilde{E}_1)_{(a)}+(\tilde{E}_2)_{(a)}) + \dim((\tilde{E}_1)_{(a)} \cap (\tilde{E}_2)_{(a)})$

$= \dim((\tilde{E}_1)_{(a)}) + \dim(\tilde{E}_2)_{(a)})$

$= (\dim(\tilde{E}_1))_{(a)} + (\dim(\tilde{E}_2)_{(a)})$

$= (\dim(\tilde{E}_1) + (\dim(\tilde{E}_2))_{(a)}$

Therefore, $\dim(\tilde{E}_1+\tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2) = \dim(\tilde{E}_1)+\dim(\tilde{E}_2)$.

Proposition 6.5:

All finite dimensional vector spaces $\tilde{E} = (E, \mu)$ have fuzzy basis.

Proof:

Since we know that all fuzzy vector spaces $\tilde{E} = (E, \mu)$ for which $\mu(E)$is upper well ordered have a fuzzy basis, we shall show that $\mu(E)$ is upper well ordered.

If $\mu(E)$ is not upper well-ordered then $\mu(E) \in [0,1]$ has an increasing monotonic limit. There exists a sequence $\{xi\}^{\infty}_{i=1}$ in E such that $\{\mu(x_i)\}^{\infty}_{i=1}$ is strictly increasing sequence with limit α.

Let $\mu(x_1) = \beta > 0$. Since $\tilde{E} = (\mu, E)$ be any fuzzy vector space and $\{Xi\}^{N}_{i=1}$ E\{0} which has distinct μ – values is linearly and fuzzy linearly independent.

Let $\tilde{E} = (E, \mu)$ be a fuzzy vector space. Then any set of vector $\{Xi\}^{N}_{i=1}$ E| {0} which has distinct μ value linearly and fuzzy linearly independent).$\{x_i\}^{\infty}_{i=1}$ is linearly independent consider the following sequence of bases for E.

Hn = the extension of the linearly independent set $\{x_i\}^{n}_{i=1}$ to basis for E.

Now clearly we have,

$\mu(x_1) < \mu(x_2) < \mu(x_3) <$ …………..

and $\mu(x_1) > \beta$

$\Longrightarrow$ $\mu(x_i) > \beta$ for all i

$\sum_{x \in Hn} \mu(x) > n\beta$

Which implies that dim $(\tilde{E}) = \infty$, which is a contradiction to the fact that $\tilde{E}$ is finite dimensional. $\mu(x) \subset [0,1]$ has no increasing monotonic limit, μ(E) is upper well ordered

By theorem 3.1 $\tilde{E} = (E, \mu)$ has a fuzzy basis. He

Lemma 6.6:

If $\tilde{E} = (E, \mu)$ is a finite dimensional vector space then for all $\alpha \in \mu(E)\backslash\{0\}$, $E^{\circ}\mu$ is finite dimensional.

Proof :

To proof that $E^{\circ}\mu$ is finite dimensional. Let us assume that $E^{\circ}\mu$ is infinite dimensional and B is a fuzzy basis for E, then $B \cap E\mu^{\alpha}$ is also infinite. Since $B \cap E\mu^{\alpha}$ is a basis for $E\mu^{\alpha}$.

Hence $\sum_{v \in E} \mu(v) \geq \sum_{v \in B \cap E_{\mu}} \alpha\, \mu(v)$

$\geq \sum_{v \in B \cap E_{\mu}} \alpha = \infty$

$\Longrightarrow \sum_{v \in B} \mu(v) = \infty \Longrightarrow \dim(E) = \infty$

Which is a contradiction, because given $\tilde{E}$ is finite dimensional if $\dim(\tilde{E}) < \infty$

$\dim(\tilde{E}) < \infty$.E μ° must be finite dimension.

Theorem 6.7:

If $\tilde{E} = (E, \mu)$ is a finite dimensional,

$\dim(\tilde{E}) = \sum\mu(v)$, where B is any fuzzy basis for E.

Proof :

It is sufficient to prove that

$\sum_{v \in B} \mu(v) \leq \sum_{v \in B} \mu(v)$ where B* is any basis for E

By lemma 5.1 If $\tilde{E} = (E, \mu)$ is a finite dimensional vector space then for all

$\alpha \in \mu(E)\setminus\{0\}$, $E^{\circ}\mu$ is finite dimensional.for all

$\alpha > 0$, $E\mu^{\alpha}$ is a finite dimensional and B is an

$B \cap E\mu^{\alpha}$ is a fuzzy basis for $E_{\alpha} = E\mu^{\alpha}\ \mu\setminus E\mu^{\alpha}$). As $E\mu^{\alpha} \cap B$ is an independent subset of $E\mu^{\alpha}$.

we know that, $\sum_{v \in B \cap E_{\mu}} \alpha\, \mu(v) \leq \sum_{v \in B \cap E_{\mu}} \alpha\, \mu(v)$

This is true for all $\alpha > 0$, and thus we must have,

$\sum_{v \in B} \mu(v) \leq \sum \alpha\, \mu(v)$

VII. CONCLUSION

We study the relation between vector space and fuzzy bases with examples for each concept, the general properties of fuzzy bases, fuzzy dimension, and fuzzy vector space. We prove that all finite dimensional vector spaces have fuzzy bases. Also it is proved that the two fuzzy vector spaces have the same fuzzy dimensions, and the summation of two fuzzy vector spaces is equal to the summation of fuzzy dimensions.

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