Approximation of Conjugate of Lip (ξ(t), r) Function by Almost (N, p_n) Summability Method

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Abstract- *In this paper, a theorem on degree of approximation of Conjugate of Lip (ξ(t), r) Function by Almost (N, p_n) Summability Method has been established.*

Keywords- Degree of approximation, , Lipschitz space, Fourier series, Almost (*N*, l) means, (*N*, l) means.

I. INTRODUCTION

Almost convergence of a bounded sequence was first defined by Lorentz (1948). After definition of almost summability method, Qureshi (1981), Singh et al. (1995), Lal et al. (2003), determine the degree of approximation of function by almost N$\ddot{o}$rlund summability method. Nema (1992), find the degree of approximation of function by almost Euler means, Lal (2002), and Lal et al. (2006), determine the degree of approximation by almost matrix summability in weighted $(L^p, \xi(t))$ and $(\xi(t), p)$ class respectively.

II. DEFINITION

A bounded sequence $\{s_n\}$ is said to be the almost convergent to a limit s if

$$lim_{n\to\infty} \frac{1}{n+1} \sum_{v=m}^{n+m} s_v = s \tag{2.1}$$

uniformly with respect to m.

Let $\{P_n\}$ be the a sequence of non-zero real constant such that

$P_n = p_0 + p_1 + p_2 + \cdots . + p_n, \qquad P_{-1} = p_{-1} = 0$

The sequence $\{s_n\}$ is said to be the almost N$\ddot{o}$rlund (*N*, p_n) summable to s if

$$t_{n,m} = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v}\, q_v\, s_{v,m}$$

tends to s as $n \to \infty$, uniformly with respect to m, where

$$s_{v,m} = \frac{1}{v+1} \sum_{k=m}^{v+m} s_k .$$

A function $f \in Lip\, \alpha$ if

$f(x+t) - f(x) = O(t^\alpha)\,, \qquad for\ o < \alpha \le 1$.

A function $f \in Lip\,(\alpha, r)$ if

$$\left\{\int_0^{2\pi} |f(x+t) - f(x)|^r\, dx\right\}^{\frac{1}{r}} = O(t^\alpha) \quad o < \alpha \le 1,\ r \ge 1$$

.

A positive increasing function $\xi(t)$ and an integer $r > 1$ then

$$f \in\ Lip\ (\xi(t), r)$$

if

$$\left\{\int_0^{2\pi} |f(x+t) - f(x)|^r\, dx\right\}^{\frac{1}{r}} = O(\xi(t)) .$$

We shall use the following notation:

$$\psi(t) = f(x+t) - f(x-t).$$

III. KNOWN RESULTS

Singh et al. (1995) find the degree of approximation by almost Norlund summability of conjugate series of a Fourier series and prove the following theorem.

Theorem Let $\{P_n\}$ be a real non – negative monotonic non – negative sequence of coefficient $\{p_n\}$ such that $P_n \to \infty$ as $n \to \infty$.

If

$$\psi(t) = \int_0^t \psi(t)\, du = O\left[\frac{p_t}{P_t}\right] \quad as \quad t \to +0$$

and $\int_{\frac{1}{n+m}}^{\frac{1}{(n+m)\delta}} \frac{|\psi(t)|}{u}\, du = O(1)$ as $n \to \infty$, where $0 < \delta < 1$, uniformly with respect to m then the conjugate Fourier series is almost (N, p_n) is summable to $\frac{1}{2\pi}\int_0^\pi \psi(t) \cot\frac{1}{2}\, t\; dt$ at every point where this integral exists.

Lal et al. (2003) proved the following theorem.

Theorem Let almost (*N,p,q*) be a regular almost generalized Nörlund method defined by two real non – negative monotonic, non – increasing sequence of coefficient $\{p_n\}$ and $\{q_n\}$ such that

$$R_n = \sum_{v=0}^{n} p_n\; q_{n-v} \to \infty \text{ as } n \to \infty.$$

$$\log(n+m) = O(T_n) \text{ as } n \to \infty$$

uniformaly with respect to m.

If $\psi(t) = \int_0^t \psi(t)\, du = O\left[\frac{t}{R\left(\frac{1}{t}\right)}\right] \quad as \quad t \to 0$

uniformaly with respect to m, then the conjugate series of Fourier series is almost *(N,p,q)* summable to to $\frac{1}{2\pi}\int_0^\pi \psi(t) \cot\frac{1}{2}\, t\; dt$ at every point at which the integral exists in Lebesgue sense.

IV. MAIN THEOREM

If $f : R \to R$ is 2π – periodic, Lebesgue integrable and belonging to $Lip\,(\xi(t), r)$ class, then the degree of approximation of *f* by almost *(N,* p_n*)* means of its conjugate Fourier series is given by

$$\|(N,p_n) - \bar{f}\|_r = \left(\xi\left(\frac{1}{n+1}\right)\, (n+1)^{\frac{1}{r}}\right),$$

provided $\xi(t)$ satisfy the following condition

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t\,\psi(t)}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{(n+1)}\right)$$

(4.1)

$$\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}\,\psi(t)}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left((n+1)^{\delta}\right),$$

(4.2)

where δ is an arbitrary number such that $s(1-\delta) - 1 > 0$, $\frac{1}{r} + \frac{1}{s} = 1$, $1 \le r \le \infty$ and above condition holds uniformly in *x*.

V. LEMMAS

For the proof of main theorem, following Lemmas are required.

Lemma 5.1 $N_{n,m}(t) = O\left(\frac{1}{t}\right) \quad for \quad 0 < t < \frac{1}{n+1}$

Proof -

$$|N_{n,m}(t)| = \left|\frac{1}{2\pi P_n} \sum_{v=0}^{n} p_{n-v}\, \frac{\cos(v+2m+1)\frac{t}{2}\, \sin(v+1)\frac{t}{2}}{(v+1)\sin^2\frac{t}{2}}\right|$$

$$\le \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}\left|\frac{\cos(v+2m+1)\frac{t}{2}\,(v+1)\sin\frac{t}{2}}{(v+1)\sin^2\frac{t}{2}}\right|$$

$$= \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}\left|\frac{\cos(v+2m+1)\frac{t}{2}}{\sin\frac{t}{2}}\right|$$

$$\le \frac{1}{2tP_n}\sum_{v=0}^{n} p_{n-v}$$

$$= O\left(\frac{1}{t}\right).$$

Lemma 5.2 $N_{n,m}(t) = O\left(\frac{1}{(n+1)\,t^2}\right)$ $for \quad \frac{1}{n+1} < t < \pi$

Proof -

$$|N_{n,m}(t)| = \left|\frac{1}{2\pi P_n} \sum_{v=0}^{n} p_{n-v}\, \frac{\cos(v+2m+1)\frac{t}{2}\, \sin(v+1)\frac{t}{2}}{(v+1)\sin^2\frac{t}{2}}\right|$$

$$\leq \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}\left|\frac{\cos(v+2m+1)\frac{t}{2}\ \ \sin(v+1)\frac{t}{2}}{(v+1)\ \sin^2\frac{t}{2}}\right|$$

$$\leq \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}\left|\frac{1}{(v+1)\ \sin^2\frac{t}{2}}\right|$$

$$\leq \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}\left(\frac{\pi}{t}\right)^2\frac{1}{(v+1)}$$

$$= \frac{\pi}{2t^2 P_n}\sum_{v=0}^{n} p_{n-v}\ \frac{1}{(v+1)}$$

$$= \left(\frac{1}{(n+1)t^2}\right)$$

VI. PROOF OF THE THEOREM 4

Let $s_k(x)$ denote the partial sum of the conjugate Fourier series is written as

$$s_k(x) = \frac{1}{2\pi}\int_0^{\pi}\frac{\cos\left(k+\frac{1}{2}\right)t\ -\ \cos\left(\frac{t}{2}\right)}{\sin\frac{t}{2}}\ \psi(t)dt$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\psi(t)dt - \frac{1}{2\pi}\int_0^{\pi}\cot\left(\frac{t}{2}\right)\psi(t)dt$$

We have $s_{v,m} = \frac{1}{v+1}\sum_{k=m}^{v+m} s_k(x)$

$$= \frac{1}{v+1}\sum_{k=m}^{v+m}\left\{\frac{1}{2\pi}\int_0^{\pi}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\psi(t)dt - \frac{1}{2\pi}\int_0^{\pi}\cot\left(\frac{t}{2}\right)\psi(t)dt\right\}$$

We know that

$$t_{n,m}^{\cdot} = \frac{1}{P_n}\sum_{v=0}^{n} p_{n-v}\ s_{v,m}$$

$$= \frac{1}{P_n}\sum_{v=0}^{n} p_{n-v}\left[\frac{1}{v+1}\sum_{k=m}^{v+m}\left\{\frac{1}{2\pi}\int_0^{\pi}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\psi(t)dt - \frac{1}{2\pi}\int_0^{\pi}\cot\left(\frac{t}{2}\right)\psi(t)dt\right\}\right]$$

$$t_{n,m}^{\cdot} - \left(-\frac{1}{2\pi}\int_0^{\pi}\cot\left(\frac{t}{2}\right)\psi(t)dt\right) = \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}\frac{1}{v+1}\sum_{k=m}^{v+m}\int_0^{\pi}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\psi(t)dt$$

$$t_{n,m}^{\cdot} - \tilde{f}(x) = \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}\int_0^{\pi}\frac{\sin(v+m+1)t - \sin mt}{2(v+1)\sin^2\frac{t}{2}}\psi(t)dt$$

$$= \frac{1}{2\pi P_n}\sum_{v=0}^{n} p_{n-v}\int_0^{\pi}\frac{\cos(v+2m+1)\frac{t}{2}\ \sin(v+1)\frac{t}{2}}{(v+1)\sin^2\frac{t}{2}}\psi(t)dt$$

$$= \int_0^{\pi} N_{n,m}(t)\ \psi(t)\ dt$$

$$= \left\{\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}.\right\} N_{n,m}(t)\ \ \psi(t)\ dt$$

$$= I_1 + I_2 \text{ .(say)} \tag{6.1}$$

Applyiny Hölder inequality and the fact $\psi(t) \in Lip\ (\xi(t), r)$, we get

$$I_1 = \int_0^{\frac{1}{n+1}} |N_{n,m}(t)||\ \psi(t)|\ dt$$

$$= \left\{\int_0^{\frac{1}{n+1}}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^r dt\right\}^{\frac{1}{r}}\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)N_{n,m}(t)}{t}\right)^s dt\right\}^{\frac{1}{s}}$$

$$= O\left(\frac{1}{n+1}\right)\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)N_{n,m}(t)}{t}\right)^s\ dt\right\}^{\frac{1}{s}}$$

$$= O\left(\frac{1}{n+1}\right)\left\{\int_0^{\frac{1}{n+1}}\left(\frac{\xi(t)}{t}\left(\frac{1}{t}\right)\right)^s\ dt\right\}^{\frac{1}{s}}$$

, by Lemma (5.1)

$$= O\left(\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right)\left\{\frac{t^{-2s+1}}{(-2s+1)}\right\}^{\frac{1}{s}}$$

$$= O\left(\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right) O\left(\frac{1}{(n+1)^{-2s+1}}\right)^{\frac{1}{s}}$$

$$= O\left[\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\frac{1}{(n+1)^{-2+\frac{1}{s}}}\right]$$

$$= O\left[(n+1)^{1-\frac{1}{s}}\ \ \xi\left(\frac{1}{n+1}\right)\right]$$

$$= O\left[(n+1)^{\frac{1}{r}}\ \ \xi\left(\frac{1}{n+1}\right)\right]. \tag{6.2}$$

Now,

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} |N_{n,m}(t)||\ \psi(t)|\ dt$$

$$=\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{t^{-\delta}\,|\psi(t)|}{\xi(t)}\right)^{r}dt\right\}^{\frac{1}{r}}\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)N_{n,m}(t)}{t^{-\delta}}\right)^{s}dt\right\}^{\frac{1}{s}}$$

$$=O\left((n+1)^{\delta}\right)\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)N_{n,m}(t)}{t^{-\delta}}\right)^{s}dt\right\}^{\frac{1}{s}},$$

$$=O\left((n+1)^{\delta}\right)\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta}}O\left(\frac{1}{(n+1)t^{2}}\right)\right)^{s}dt\right\}^{\frac{1}{s}},$$

by Lemma (5.2)

$$=O\left((n+1)^{\delta-1}\right)\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+2}}\right)^{s}dt\right\}^{\frac{1}{s}}$$

$$=O\left((n+1)^{\delta-1}\right)\left\{\int_{\frac{1}{n+1}}^{\pi}\left(\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-2}}\right)^{s}\frac{dy}{y^{2}}\right\}^{\frac{1}{s}}$$

$$=O\left((n+1)^{\delta-1}\right)\left\{O\xi\left(\frac{1}{n+1}\right)\int_{\frac{1}{n+1}}^{\pi}\left(\frac{1}{y^{\delta-2}}\right)^{s}\frac{dy}{y^{2}}\right\}^{\frac{1}{s}}$$

$$=O\left((n+1)^{\delta-1}\ \xi\left(\frac{1}{n+1}\right)\right)\left(\int_{\frac{1}{n+1}}^{\pi}y^{-(\delta-2)s-2}\,dy\right)^{\frac{1}{s}}$$

$$=O\left((n+1)^{\delta-1}\ \xi\left(\frac{1}{n+1}\right)\right)\left((n+1)^{-\delta+2-\frac{1}{s}}\right)$$

$$=O\left(\ \xi\left(\frac{1}{n+1}\right)(n+1)^{1-\frac{1}{s}}\right)$$

$$=O\left(\ \xi\left(\frac{1}{n+1}\right)(n+1)^{\frac{1}{r}}\right). \qquad (6.3)$$

From (6.1) (6.2) and (6.3) we get

$$\left|t_{n,m}^{p,q}-\tilde{f}(x)\right|=O\left(\ \xi\left(\frac{1}{n+1}\right)(n+1)^{\frac{1}{r}}\right)$$

$$\left\|t_{n,m}^{p,q}-\tilde{f}(x)\right\|_{r}=\left[\int_{0}^{2\pi}\left\{(n+1)^{\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)^{r}\right\}^{\frac{1}{r}}dx\right]$$

$$=O\left[\int_{0}^{2\pi}\left\{(n+1)^{\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)^{r}\right\}^{\frac{1}{r}}dx\right]$$

$$=O\left[(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\left\{\int_{0}^{2\pi}dx\right\}^{\frac{1}{r}}\right]$$

$$=O\left[(n+1)^{\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right].$$

This completes the proof of the theorem.

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