Design Efficiency Analysis of a Minimum Length Supersonic Nozzle Derived Using MOC

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Abstract- *This project presents a comprehensive analysis of the design efficiency of a minimum length supersonic nozzle, derived using the Method of Characteristics (MOC). The study focuses on developing a nozzle profile with minimum lengthensuring efficient expansion of supersonic flows to the desired exit conditions. The MOC is employed to generate the nozzle contour, which is subsequently evaluated using Computational Fluid Dynamics (CFD) simulations to validate the design's performance.The nozzle's design efficiency is assessed based on the exit Mach number consistency.The results indicate that the MOC-derived nozzle achieves significant potentials without compromising on performance, offering possible benefits for high-speed flow applications.*

I. INTRODUCTION

Supersonic convergent-divergent (CD) nozzles are critical components in various high-speed aerospace applications, including rocket engines, jet engines, and supersonic wind tunnels. These nozzles facilitate the acceleration of gases from subsonic to supersonic speeds, enabling the efficient conversion of thermal energy into kinetic energy. The design of CD nozzles is pivotal in determining the overall performance and efficiency of propulsion systems, particularly in supersonic and hypersonic regimes.

The operation of a supersonic nozzle involves the compression of flow through a converging section, followed by a rapid expansion in the diverging section, where the flow reaches and often exceeds the speed of sound. The precise contour of the nozzle plays a crucial role in ensuring the uniformity of the flow at the exit, minimizing shock waves, and achieving the desired exit Mach number. Inaccuracies in the nozzle design can lead to flow separation, shock-induced losses, and reduced thrust, undermining the overall efficiency of the propulsion system.Given the importance of optimized nozzle design, various methods have been developed to generate nozzle contours that maximize performance. Among these, the Method of Characteristics (MOC) stands out as a powerful analytical tool for designing supersonic nozzles. MOC allows for the systematic generation of nozzle shapes that ensure smooth and shock-free expansion of the flow, making it a preferred approach for designing nozzles that require high precision.

This project focuses on the design and efficiency analysis of a minimum length supersonic nozzle derived using MOC. The goal is to create a nozzle that achieves the required supersonic exit conditions with the shortest possible length. The design generated using MOC is further validated through Computational Fluid Dynamics (CFD) simulations, which provide detailed insights into the flow characteristics and performance metrics of the nozzle.

SCOPE

1. The primary objective of this project is to derive coordinatesfor a minimum length supersonic convergent-divergent (CD) nozzle capable of achieving an exit Mach number of 3 using the Method of Characteristics (MOC).
2. To design contours (CAD Model) for the derived coordinates of Mach number 3.
3. To validate thedesign efficiency of the obtained contours using fluent (CFD).

II. THEORY AND METHODOLOGY

Supersonic flows, characterized by flow velocities exceeding the speed of sound, exhibit unique fluid dynamics compared to subsonic flows. In supersonic regimes, disturbances travel downstream, and flow behavior is governed by compressibility effects, leading to phenomena such as shock waves and expansion fans. To achieve and maintain these high-speed conditions, convergent-divergent (CD) nozzles are essential.

The CD nozzle operates by first compressing the flow in the convergent section, increasing pressure and velocity until the flow reaches Mach 1 at the throat—the narrowest point of the nozzle. Beyond the throat, the diverging section allows the flow to expand, accelerating it to supersonic speeds. The precise contour of the nozzle is critical to ensure that the expansion is smooth and controlled, avoiding unwanted shockwaves that could disrupt the flow and reduce efficiency. Such that to design a precise contour of divergent section of CD nozzle, Method of Characteristics (MOC) is widely used.

Method of Characteristics (MOC):

The Method of Characteristics (MOC) is a powerful analytical tool used to solve hyperbolic partial differential equations, which describe the behavior of supersonic flows in a convergent-divergent nozzle. MOC provides a systematic approach to designing nozzles by generating a characteristic mesh that ensures the smooth and shock-free expansion of the flow. The MOC is based on the governing equations of inviscid, steady, and compressible flow, specifically the conservation of mass, momentum, and energy. These equations can be simplified under the assumption of isentropic flow, resulting in the following system of partial differential equations in two-dimensional space:

1. **Continuity Equation:**

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0$$

where ρ is the density, and u and v are the velocity components in the x and y directions, respectively.

2. **Momentum Equations:**

$$\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

where p is the pressure.

3. **Energy Equation:**

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = 0$$

where T is the temperature.

In supersonic flow, these equations simplify along specific directions, known as characteristic lines, where the information propagates. For isentropic, inviscid flows, the characteristic equations can be expressed as:

$$\frac{dy}{dx} = \tan(\theta \pm \mu)$$

where θ is the flow angle, and μis the Mach angle, given by:

$$\mu = \sin^{-1}\left(\frac{1}{M}\right)$$

where M is the local Mach number.

The characteristic lines, labeled as C+ and C−, represent paths along which the flow properties can be integrated and calculated. These characteristics intersect, allowing the determination of flow variables at nodal points.

Deriving Coordinates of Minimum Length Nozzle:

Step by step coordinates deriving process of the minimum length nozzle for an exit Mach number 3.0 will be performed.

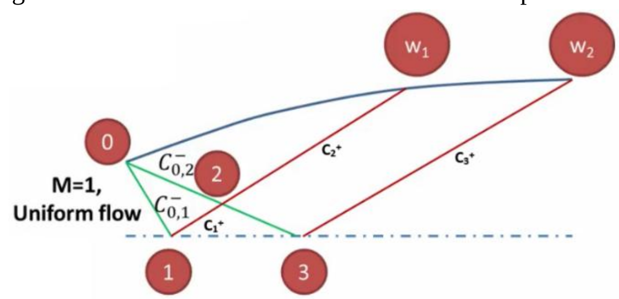

Fig. 1. Representation of Characteristic lines in the divergent section.

- Considering 2 characteristics originating at point 0.
- The purposeis to calculate the coordinates of points W_1, W_2.
- As part of the procedure, field in points 1,2,3 will be calculated.
- Automation of the process with a good number of characteristics (for e.g. 100-200, can be more if exit Mach number is large) to be used to obtain well-resolved nozzle wall contour.
- The subsonic convergent part of nozzle is usually designed using a smooth contour like straight line with specific inclination angle, an arc or a higher-order polynomial.
- In this case, flow at the throat is considered to be uniform.
- In some cases, a source flow solution is obtained to get a more realistic velocity distribution at the throat.
- Here, the half-throat height is taken as 1, the results can be then scaled to an appropriate throat height.

At point 0, Prandtl-Meyer expansion is considered;

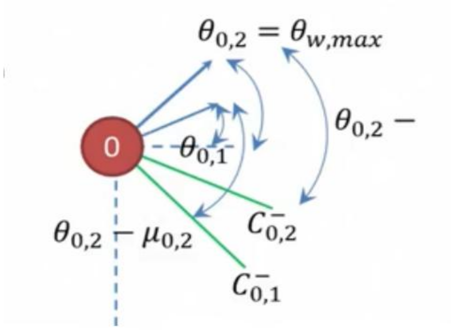

Fig. 2. Representation of point 0.

Let point 0 be (0,1),
$\Rightarrow \theta_{Wmax} = \nu/2$
$\Rightarrow M_e = 3$
From Isentropic Table, $\nu = 49.75731$
$\Rightarrow \theta_{Wmax} = 24.88$
$\Rightarrow \theta_{(0,1)} = 12.44 \quad \Rightarrow \nu_{(0,1)} = 12.44$
$\Rightarrow \theta_{(0,2)} = 24.88 \quad \Rightarrow \nu_{(0,2)} = 24.88$
$\therefore \Delta\theta = \nu_{2-}\nu_1$
$\Rightarrow M=1, \nu_{1=0}$
$\therefore \Delta\theta = \nu_2 \& \nu_{2=} \theta$ (or) θ_2

Points	X	Y	θ	ν	M	μ	C+ dy/dx	C− dy/dx	K+	K-
0, 1	0	1	12.44	12.44	1.52	41.13955	-	-	0	24.88
0, 2	0	1	24.88	24.88	1.52	30.85192	-	-	0	49.75

Table 1. Point 0 Coordinates.

Point 1:(Intersection point of $C_{0,1}^{-}$ with line of symmetry)

$\Rightarrow \theta_1 = 0,\ \theta_1+\nu_1 = K_{0,1}^{-} = 24.88$
$\Rightarrow \nu_{1=} 24.88$
$\therefore M_1 = 1.95,\ \mu_1 = 30.85192$

Equation for line joining 0&1,

$\Rightarrow y_1 - y_2 / x_1 - x_2 = \tan(\theta - \mu)$
$\Rightarrow 0-1/x_1-0 = \tan[(12.44+0)/2 - (41.13955+30.8519)/2]$
$\therefore x_{1=} 1.7478$

Point	X	Y	θ	ν	M	μ	C+dy/dx	C− dy/dx	K+	K-
1	1.7478	0	0	24.88	1.95	30.85192	-	-0.572	-24.88	24.88

Table 2. Point 1 Coordinates.

Point 2:(Interior point at intersection of $C_{0,2}^{-}$ & C_1^{+})

$\Rightarrow K_1^{+} = \theta_2 - \nu_2 = -24.88$
$\Rightarrow K_{0,2}^{-} = \theta_2 + \nu_2 = 49.757$
$\therefore \theta_2 = 12.44,\ \nu_2 = 37.317$
$M_2 = 2.43,\ \mu_2 = 24.3$
Equation for line joining 0&2,
$[1\text{-}2]; y_2 - 0/ x_2 - 1.7478 = \tan[(0+12.44)/2 + (30.85+24.3)/2]$
$\Rightarrow y_2 = 0.6693x_2 - 1.169829$ **1**
$[0\text{-}2]; y_2 - 1/ x_2 - 0 = \tan[(24.88+12.44)/2 - (30.85+24.3)/2]$
$\Rightarrow y_2 = -0.1568x_2 + 1$ **2**
Equation**1 – 2**;$x_2 = 2.6265,\ y_2 = 0.5882$

Point 2	X	Y	θ	ν	M	μ	C+dy/dx	C− dy/dx	K+	K-
	2.6265	0.5882	12.44	37.317	2.43	24.3	0.6693	-0.156	-24.88	49.757

Table 3. Point 2 Coordinates.

Point W$_1$:(C_2^{+} meet walls at W_1)
$\Rightarrow \theta_{W1} = \theta_2 = 12.44$ [No wall reflection or cancellation]
$[W_1, 2]; y_{w1} - 0.5882 / x_{w1} - 2.6265 = \tan(12.44 + 24.3)$
$\Rightarrow y_{w1} = 0.7464x_{w1} - 1.37237$ **3**
$[W_1, 0]; y_{w1} - 1 / x_{w1} - 0 = \tan[(24.88 + 12.44) / 2]$
$\Rightarrow y_{w1} = 0.337x_{w1} + 1$ **4**
Equation **3 – 4**; $x_{w1} = 5.7947,\ y_{w1} = 2.9528$

Point W$_1$	X	Y	θ	ν	M	μ	C+dy/dx	C− dy/dx	K+	K-
	5.7947	2.9528	12.44	37.317	2.43	24.3	0.746	0.337	-24.88	49.757

Table 4. PointW1 Coordinates.

Point 3:(Intersection of C_2^- with the symmetry line)

$\therefore \theta_3 = 0, \nu_2 = 49.757$

$M_3 = 3$

$\mu_3 = 19.47124$

Equation for line joining 2&3,

$\Rightarrow 0 – 0.5882/ x_3 – 2.6265 = \tan [(12.44+0)/2 - (24.3+19.47)/2]$

$x_3 = 4.7242$

Point 3	X	Y	θ	ν	M	μ	C+dy/dx	C−dy/dx	K+	K-
	4.7242	0	0	49.757	3	19.47	-	-0.28	-49.75	49.75

Table 5. Point3 Coordinates.

Point W_2: (C_3^+ meet walls at W_2)

$\therefore \theta_3 = \theta_{w2} = 0$

$[W_2, 3]$; $y_{w2} – 0 / x_{w2} – 4.724 = \tan (0+19.47)$

$\Rightarrow y_{w2} = 0.3535x_{w2} – 1.67$ **5**

$[W_2, 0]$; $y_{w2} – 2.9528 / x_{w2} – 5.7947 = \tan [(12.44+0)/2]$

$\Rightarrow y_{w2} = 0.10898x_{w2}+2.3213$ **6**

Equation**5 – 6**; $x_{w2} = 16.323$, $\mathbf{y_{w2} = 4.10018}$

$\therefore$From Isentropic tablefor M=3, $\mathbf{A_e/A_t = 4.23456}$. Since it was taken only 2 characteristic lines, getting$\mathbf{A_e/A_t = 4.10018}$. Accuracy can be increased by taking a large number of characteristic lines.

Here, the obtained coordinates are;

Coordinates	X	Y
Point 0	0	1
Point W_1	5.7947	2.9528
Point W_2	16.323	4.10018

Table 6. MOC Derived Coordinates.

CAD Model:

For the preparation of CAD model (Surface, 3D Fluid domain), the derived coordinates are scaled to;

Coordinates	X	Y
Point 0	0	0.5
Point W_1	2.89735	1.4764
Point W_2	8.1615	2.05009

Table 7. Scaled Coordinates.

Nozzle Contour design parameters;

- Nozzle Total Length = 14mm
- Length ofConvergent Section = 5.8385mm
- Length ofDivergent Section = 8.1615mm
- Half Cone Angle of Convergent Section = 15°
- Length of Extended Domain = 140mm

The nozzle contours are designed as 2D Surface, 2D Surface with domain extension and 3D fluid domain to enquire the design accuracy of the derived coordinates.

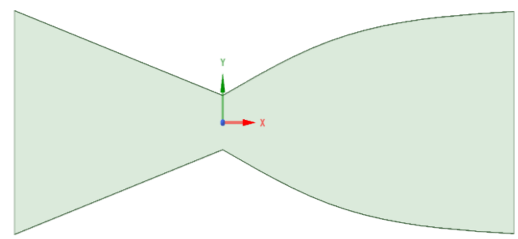

Fig. 3. M=3 2D Surface Contour

Fig. 4.

M=3 2D Surface Contour with Domain Extension

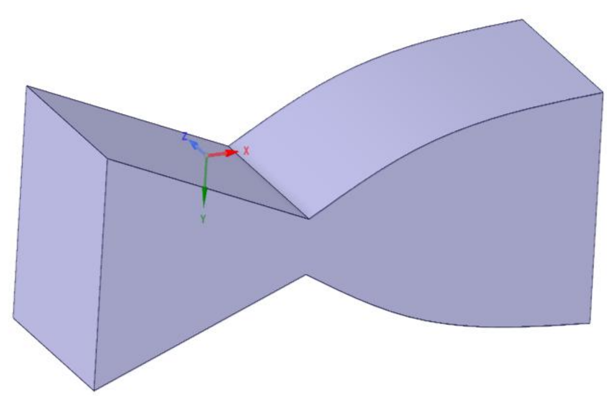

Fig.5.M=3 3D Fluid Domain Contour

CFD Setup and Simulation:

For Inlet boundary conditions, the Isentropic relations are used for the calculation of inlet pressurewith reference to Exit/Design Mach number.

$$\frac{p_0}{p_1} = \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\gamma/\gamma-1}$$

□P_0 is the inlet pressure.
□P_1 is the ambient exit pressure.
□ γ is the ratio of specific heats (usually γ=1.4 for air).
□M_1 is the Mach numberat the exit.
Here;
M_1 = 3, P_1 = 101325pa
$\Rightarrow P_0 = 36.8\ P_1$
$\therefore P_0$ = 3728760 ≈ 38 bar
&T_0 = 300 K(Standard)

Pre-processing &Boundary Conditions:

Solver Type	Pressure Based
Turbulence Model	Viscous (SST K-Omega)
Inlet	Pressure inlet
Outlet	Pressure outlet
Solution Method	Coupled

Table 8. Boundary Conditions.

Solution Convergence Plot:

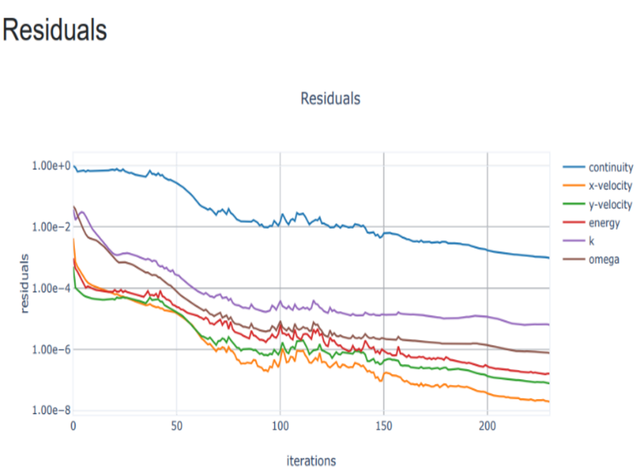

Fig. 6. Residuals Convergence Plot, M=3 2D Surface

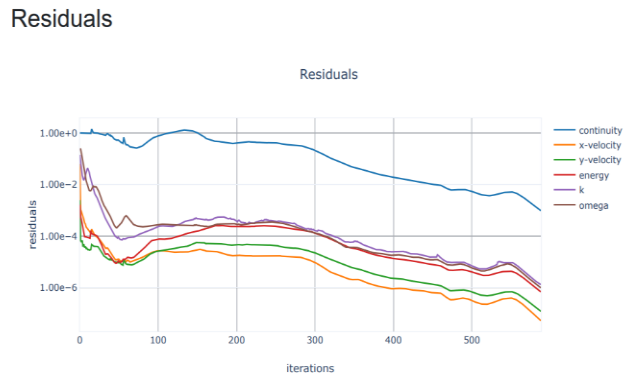

Fig. 7. Residuals Convergence Plot, M=3 2D Surface with Extended Domain

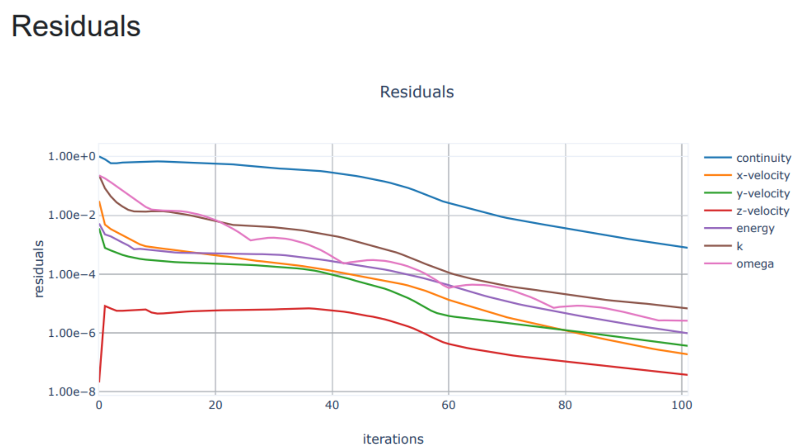

Fig. 8. Residuals Convergence Plot, M=3 3D Fluid Domain

Post Processing:

1. 2D Surface Result contours;

- **Mach NumberContour**

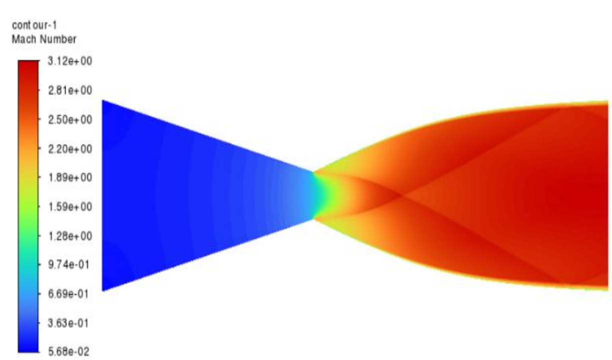

Fig. 9. 2D Surface Mach Number Contour

- **Pressure Contour**

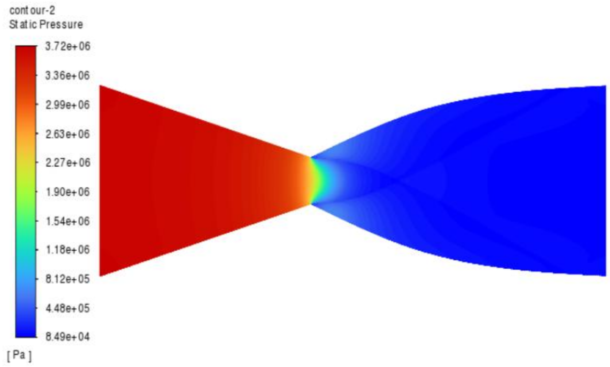

Fig. 10. 2D Surface Pressure Contour

- **Temperature Contour**

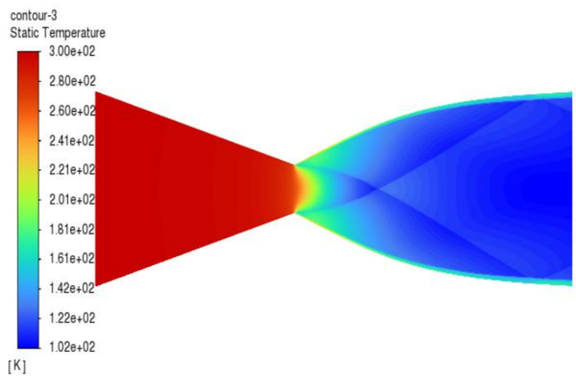

Fig. 11. 2D Surface Temperature Contour

2. 3D Fluid Domain Result Contours;

- **Mach Number Contour**

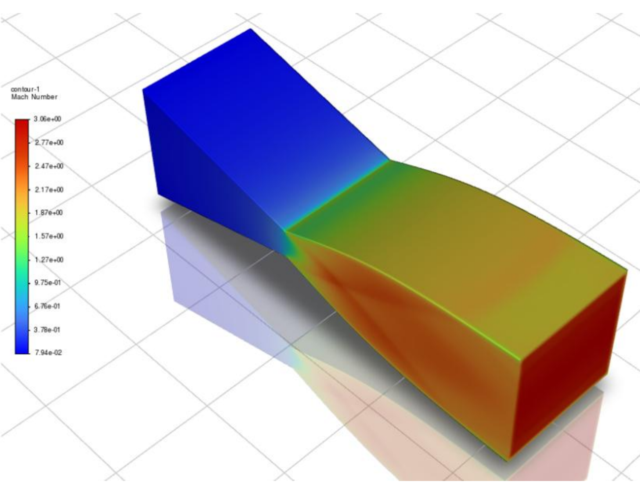

Fig. 12. 3D Fluid Domain Mach Number Contour

- **Pressure Contour**

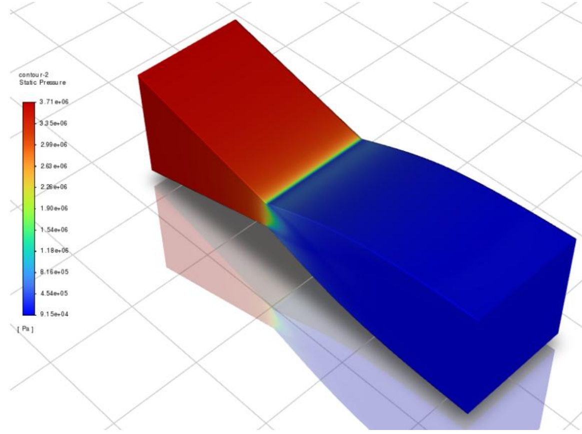

Fig. 13. 3D Fluid Domain Pressure Contour

- **Temperature Contour**

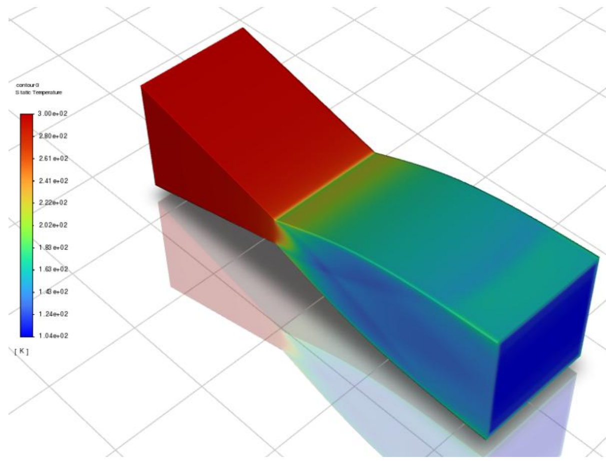

Fig. 14. 3D Fluid Domain Temperature Contour

3. Supersonic Flow Regime;

2D Surface Extended Domain Mach Contour

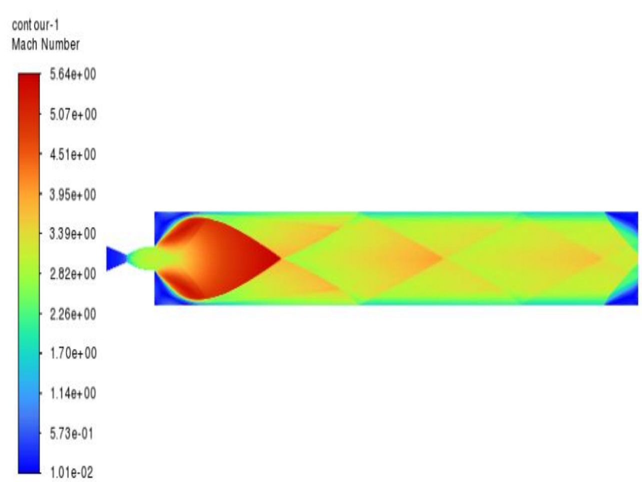

Fig. 15. 2D Surface Extended Domain Mach Contour

III. RESULT AND CONCLUSION

Design Mach Number	3.0
2D Analysis Exit Mach Number	2.96
3D Analysis Exit Mach Number	2.91

Table 9. Results.

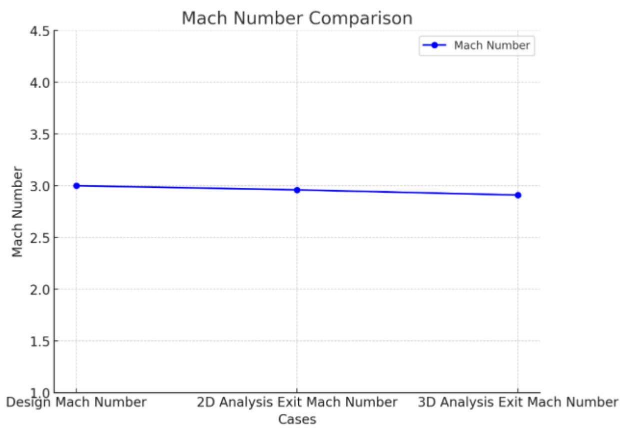

Fig. 16. Plot: Result Cases vs Mach Number

The final CFD simulations provided valuable insights into the nozzle's performance. The 2D CFD analysis resulted in an exit Mach number of **2.96**, closely matching with the design target and demonstrating the effectiveness of the MOC in generating a nozzle contour that supports efficient supersonic expansion. The slight deviation from the target Mach number indicates minimal losses andit can be due to slight variation in the area ratio of derived $\mathbf{A_e/A_t}$ **= 4.10018**andwhich isneed to bethe value of $\mathbf{A_e/A_t}$ **= 4.23456**forMach number 3. Accuracy of area ratio can be increased by taking alarge number of characteristic lines.Hence, confirming that the MOC-based design approach is robust and reliable for 2D configurations.

In the 3D CFD analysis, the exit Mach number was found to be **2.91**. While slightly lower than the 2D result, this outcome reflects the additional complexities introduced in three-dimensional flows, such as potential boundary layer effects and three-dimensional expansion phenomena. Despite these challenges, the MOC-derived design still achieved a high level of performance, indicating that the nozzle is well-suited for practical supersonic applications.

Overall, the project successfully demonstrated the capability of the MOC to design efficient, minimum-length supersonic nozzles. The CFD validation confirmed that the designed nozzle performs effectively under both 2D and 3D conditions, with only minor deviations from the desired Mach number. These findings support the continued use of MOC in the design of advanced supersonic nozzles, contributing to the development of more compact and efficient high-speed flow systems.

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