On N(k)-Quasi Einstein Manifold

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Abstract- *In this N(k)-Quasi Einstein Manifolds (N(k)-(QE)n) are introduced and we have studied an N(k)-quasi Einstein manifold satisfying R(ξ, X).Ã , where* $\tilde{A}$ *is the pseudo-projective curvature.*

Keywords- k-nullity distribution, Killing vector field, N(k)-quasi Einstein manifold, Pseudo-projective curvature tensor, Quasi-Einstein manifold

I. INTRODUCTION

A Quasi-Einstein manifold is precisely a simple and natural generalization of a Einstein manifold. A quasi-Einstein manifold is defined as a non -flat Riemannian manifold (M_n ,g) (n>2) [2] if the Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

(1.1) S(X,Y) =a g(X,Y) + b η(X) η(Y),
X,Y∊TM

where $b \neq 0$,*a* and *b* are smooth functions on the manifold and *η* is a not vanishing of 1-form such that its Ricci operator Q satisfies

(1.2) Q= a*I* + bn $\otimes$ ξ

for some smooth functions a and b ≠ 0, where η is a non-zero 1-form such that,

(1.3) g(X, ξ) =η(X); g(ξ, ξ) =η(ξ) = 1

for the associated vector field *ξ*. The generator of the manifold is the unit vector field *ξ*. Where *a* and *b* are scalars and are known as the associated scalars.

$(QE)_n$ is the notation for an n-dimensional manifold of this kind . If b= 0 and a= $\frac{r}{n}$ then this reduces to the well-known Einstein manifold. This suggest the name given to this manifold as 'Quasi-Einstein Manifold'.

In an n-dimensional quasi-Einstein manifold the Ricci tensor has precisely two distinct eigen values a and a + b , where multiplicity of a is (n -1) and a + b is simple. A natural example of a quasi-Einstein manifold is proper η-Einstein contact metric manifold ([1], [3]).

In 2007 Mukut Mani Tripathi and Jeong-Sik Kim, it is proved that conformally flat quasi Einstein manifolds are Certain N(k)-quasi Einstein manifolds.

Definition 1. *Let (Mn, g) be a non flat Riemannian manifold. If the ricci tensor S of (Mn, g) isnon zero and satisfies S(X, Y) = ag(X, Y)+ bA(X)B(Y)+ cB(X)A(Y),*

where a, b and c are smooth functions and A and B are non zero 1-forms such that g(X,U) =A(X) and g(X, V) = B(X) for all vector fields X, and U and V being the orthogonal unit vectorfields called generators of the manifold belong to N(k), then we say that (Mn, g) is a N(k)-quasi Einstein manifold and is denoted by N(k)−(QE)n. [11]

M.M. Tripathi and J.S. Kim [9] in 2007 studied a quasi-Einstein manifold whose generator ξ belongs to the k-nullity distribution N(k) and called such a manifold as N(k)-quasi Einstein manifold. Conformally flat quasi-Einstein manifolds are certain N(k)-quasi Einstein manifolds is proved in [9]. In [8] the derivation conditions R(ξ,X).R = 0 and R(ξ,X).S = 0 are studied where R and S denotes the curvature and Ricci tensor, respectively. Cihan Ozgur and M.M. Tripathi [5] continued the study of the N(k)-quasi Einstein manifold.The derivation conditions Z(ξ,X).R = 0and Z(ξ,X).Z = 0 on N(k)-quasi Einstein manifold were studied in [5] by Ozgur, C., Tripathi, M. M, where Z is the concircular curvature tensor. Moreover, in [5], for an N(k)-quasi Einstein manifold it was proved that k = $\frac{a+b}{n-1}$ C. Ozgur [4], in 2008, studied the condition R.A = 0 for an N(k)-quasi Einstein manifold, where A denotes the projective curvature tensor and some physical examples of N(k)-quasi Einstein manifolds are given. Again, in 2008, C. Ozgur and Sibel Sular [6], studied N(k)-quasi Einstein manifold satisfying R(ξ,X).A = 0and R(ξ,X). Ã = 0, where A and Ã represent the Weyl conformal curvature tensor and the quasi-conformal curvature tensor, respectively. This paper is a continuation of previous studies.

The paper is organized as follows after introduction in Section II we discussed N(k)-quasi Einstein manifold. In Section III, we studied N(k)-quasi Einstein manifold satisfying R(ξ,X). Ã = 0and Section IV deals with aRicci symmetric quasi-Einstein manifold with constant associated scalars.

II. N(k)-QUASI EINSTEIN MANIFOLD

The k-nullity distribution N(k) of a Riemannian manifold Mn is defined by[8]

N(k) : p $\rightarrow$Np(k) = {Z ϵTpM | R(X, Y,Z) = k(g(Y,Z)X - g(X,Z)Y

for all X, Y ϵTM, where k is some smooth function.

The notion of $N(k)$-quasi Einstein manifold was introduced by Tripathi and Kim [9]. If the generator ξ of a quasi Einstein manifold belongs to the k nullity distribution $N(k)$ for some smooth function k, then this quasi Einstein manifold is called $N(k)$-quasi Einstein manifold [9]. The $N(k)$-quasi Einstein manifolds have also been studied by Ozgur [4], Ozgur and Sular [6],Ozgur andTripathi [5].

If the generator ξ of the quasi-Einstein manifold M_n belongs to the k-nullity distribution N(k) for some smooth function k, then M_n is called N(k)-quasi Einstein manifold [9].

On N(k)-quasi Einstein manifold, we have [9]

(2.1) $R(Y,Z)\ \xi = k(\eta(Z)Y - \eta(Y)Z)$.

The above equation is equivalent to

(2.2) $R(\xi, Y)Z = k(g(Y,Z)\ \xi - \eta(Z)Y)$.

In particular, the above two equations imply that

(2.3) $\eta(R(Y,Z)\ \xi) = 0$:

Moreover, it is known [5] that

In an n-dimensional N(k)-quasi Einstein manifold, it follows that

(2.4) $k = \frac{\alpha+\beta}{n-1}$

Theorem 2.1. *An n-dimensional conformally flat quasi Einstein manifold is an N* $(\frac{\alpha+\beta}{n-1})$ *-quasi Einstein manifold.*

Thus, we see that n-dimensional conformally flat quasi Einstein manifolds are natural examples of N(k)-quasi Einstein manifolds. It is well-known that in a 3-dimensional Riemannian manifold (M^3,g) the conformal curvature tensor vanishes. [16]

Corollary 2.2. *Each* 3*-dimensional quasi Einstein manifold is an N*$(\frac{\alpha+\beta}{2})$ *quasi Einstein manifold.*

Let (M_n, g) be an $N(k)$-quasi Einstein manifold. Then, we have

(2.5) $R(Y,Z)\ \xi = k\ (\eta\ (Z)\ Y - \eta\ (Y\ )Z)$.

The equation (2.7) is equivalent to

(2.6) $R(\xi,Y\ )Z = k\ (g\ (Y,Z)\ \xi - \eta\ (Z)\ Y\ )$.

In particular, the above equation implies that

(2.7) $R(\xi,Y\ )\ \eta = k\ (\eta\ (Y\ \xi - Y\ )$.

From (2.7) and (2.8), we have

(2.8) $\eta\ (R(Y,Z)\ \xi) = 0$,

(2.9) $\eta\ (R(\xi, Y\ )Z) = k\ (g\ (Y,Z) - \eta(Y\ )\ \eta\ (Z))$.

III. N(k)-QUASI EINSTEIN MANIFOLD SATISFYING R(ε, X). Ã = 0.

In 2002, B. Prasad [7] introduced the notion of a pseudo-projective curvature tensor. The pseudo-projective curvature tensor Ã on a manifold M_n of dimension n is defined as follows.

$\tilde{A}(X,Y\ )Z = \alpha\ R(X, Y\ )Z + \beta[S(Y,Z)X - S(X,Z)Y\ ]$

(3.1) $-\ \frac{r}{n}\ [\ \frac{\alpha}{n-1} + \beta][g(Y,Z)X - g(X,Z)Y\ ]$,

where α and β are the constants such that $\alpha, \beta \neq 0$, R is the curvature tensor and S is the Ricci tensor. It is obvious that if $\alpha = 1$ and $\beta = -\frac{1}{n-1}$, then the pseudo-projective curvature tensor reduces to a projective curvature tensor.
Let, N(k)-quasi Einstein manifold satisfy the condition

(3.2) $R(\varepsilon, Y\ ).\ \tilde{A} = 0$.

This implies

$0 = R(\xi\ , Y\ )\ \tilde{A}\ (U, V\ )Z\ -\ \tilde{A}\ (R(\xi\ , Y\ )U, V\ )Z$

(3.3) $-\ \tilde{A}\ (U,R(\xi, Y\ )V\ )Z - \tilde{A}\ (U,V\ )R(\xi\ , Y\ )Z$.

Taking inner product of the equation (3.3) with ξ, we get

$0 = g(R(\xi, Y)\tilde{A}(U,V)Z, \xi) - g(\tilde{A}(R(\xi, Y)U, V)Z, \xi) - g(\tilde{A}(U, R(\xi, Y)V)Z, \xi) - g(\tilde{A}(U,V)R(\xi, Y)Z, \xi)$.

By virtue of (2.2), the above equation gives

$$0 = k\tilde{A}(U,V,Z,Y) - \eta(\tilde{A}(U,V)Z)\eta(Y) - g(Y,U)\eta(\tilde{A}(\xi,V)Z) + \eta(U)\eta(\tilde{A}(Y,V)Z) - g(Y,V)\eta(\tilde{A}(U;\xi)Z) + \eta(V)\eta(\tilde{A}(U,Y)Z) - g(Y,Z)\eta(\tilde{A}(U;V)\xi) + \eta(Z)\eta(\tilde{A}(U,V)Y)], \quad (3.4)$$

where $\tilde{A}(U,V,Z,Y) = g(\tilde{A}(U,V)Z, Y)$.

Now, from (1.1), (2.1), (3.1), we have

(3.5) $\eta(\tilde{A}(X,Y)Z) = \lambda[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]$.

where $\lambda = [\alpha k - \frac{r}{n}(\frac{\alpha}{n-1} + \beta) - \beta\alpha]$, which, by 2.1, reduces to

$\lambda = b(\frac{n-2}{n})$

n . From (3.6), it follows that

(3.6) $\eta(\tilde{A}(X,Y)\xi) = o$,

(3.7) $\eta(\tilde{A}(\xi,Y)Z) = \lambda[g(Y,Z) - \eta(Y)\eta(Z)]$

And

(3.8) $\eta(\tilde{A}(X;\xi)Z) = \lambda[\eta(X)\eta(Z) - g(X,Z)]$.

Using (3.6), (3.7), (3.8) in (3.5), we obtain

(3.9) $0 = k[\tilde{A}(U,V,Z,Y) - \lambda(g(Y,U)g(V,Z) - g(Y,V)g(U,Z)]$,

which, due to the equation (3.1), yields

(3.10) $0 = k[\alpha R(X,Y,Z,W) + \beta fS(Y,Z)g(X,W) - S(X,Z)g(Y,W)g - \{\frac{r}{n}(\frac{\alpha}{n-1} + \beta) + \lambda\}g(g(Y,Z)g(X,W) - g(X,Z)g(Y,W))]$.

Contracting above equation (3.11) over X and W, we get

(3.11) $0 = k[S(Y,Z) - \mu g(Y,Z)]$,

where

$\mu = \frac{1}{\alpha+(n-1)\beta}[\lambda(n-1) + \frac{r}{n}\{\alpha + (n-1)\beta)\}]$.

Since the manifold under consideration is not an Einstein manifold, therefore it follows that k = 0.

Conversely, if k = 0, then in view of equation (2:2), we have $R(\xi,X) = 0$,

which gives $R(\xi,X).\tilde{A} = 0$. Thus, we have the following theorem

IV. RICCI-SYMMETRIC QUASI-EINSTEIN MANIFOLD

In this section we consider a quasi-Einstein manifold, whose associated scalars a and b are constant.

Definition 4.1. *A Riemannian manifold M_n is called a Ricci-symmetric manifold if its Ricci tensor S satisfies the condition*

(4.1) $(\nabla_X S)(Y,Z) = 0$,

where ∇ is the Levi-Civita connection of the Riemannian metric g.

Definition 4.2. *The Ricci tensor of Riemannian manifold is said to be cyclic parallel if*

(4.2) $(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0$.

Let M_n be a quasi-Einstein manifold, whose associated scalars are constant, then by differentiating (1.1) covariantly with respect to Levi-Civita connection we get

(4.3) $(\nabla_X S)(Y,Z) = b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y)]$.

If Ricci tensor of M_n is symmetric, then the equation (4.3) implies that

$$b((\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y)) = 0;$$

which on putting $Z = \xi$ gives,

(4.4) $(\nabla_X \eta)(Y) = 0$ as $b \neq 0$:

Putting Y=X in equation (4:4), we find

$$(\nabla_X \eta)(X) = 0$$

or equivalently

$$g(\nabla_X \xi, X) = 0,$$

and from (4:4), we also have

(4.5) $(\nabla_X \eta)(Y) + (\nabla_Y \eta)(X) = 0$.

Therefore, we have the following two theorems.

Theorem 4.1. If the quasi-Einstein manifold Mn with constant associated scalars is Ricci symmetric, then its generator ξ satisfies $g(\nabla_X \xi, X) = 0$.

Theorem 4.2. If the quasi-Einstein manifold Mn with constant associated scalars is Ricci symmetric, then its generator ξ is a Killing vector field.

Next, from (4:3), we get

$$\mathfrak{S}(X,Y,Z)(\nabla_X S)(Y,Z) = b[(\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y) + (\nabla_Y \eta)(Z)\eta(X) + (\nabla_Y \eta)(X)\eta(Z) + (\nabla_Z \eta)(X)\eta(Y) + (\nabla_X \eta)(Y)\eta(X)], \quad (4.6)$$

where $\mathfrak{S}(X,Y,Z)$ denotes a cyclic sum with respect to X, Y and Z.

i.e. $\mathfrak{S}(X,Y,Z)(\nabla_X S)(Y,Z) = (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X, Y)$.

If a generator of the quasi-Einstein manifold is a Killing vector, then we have the equation (4:5), which on using in (4:6), gives

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X, Y) = 0.$$

Thus, we may have the following theorem:

Theorem 4.3. If the generator of the quasi-Einstein manifold M_n with constant associated scalars is Killing, then its Ricci tensor is cyclic parallel.

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