Boundary Conformal Volume And The First Steklov Eigenvalue

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Abstract- *In this paper we give an overview results abut the boundary conformal volume of k- dimensional Riemannian manifold with nonempty boundary $\partial\Sigma$. we give an estimates for the first eigenvalue of the Dirichlet-to-Neumann map. we prove that the boundary n- conformal volume $\mathrm{Vol}_{bc}(\Sigma,n,\varphi) \le 2\pi$ and $\mathrm{Vol}_{bc}(\Sigma,n,\varphi) \le L^2/2A$ were φ is a conformal map from the surfaces Σ to the unit ball B^n in $\mathbb{R}^n$ we also showing that for $\epsilon > 0 \mathrm{Vol}_{rc}(\Sigma,n) \ge \mathrm{Vol}_{rc}(\Sigma,n+\epsilon)$.*

I. INTRODUCTION

In this paper develop a theory which we call boundary and relative conformal volume because it issimilar to the conformal volumetheory of Li and Yau [4] exceptingthat the boundarybe an essential role in this theory. Using the Gauss-Bonnet Theorem with boundary weshow (Theorem 2.2) that when $k = 2$, a free boundary solution has boundary length whichis a maximum over the boundary lengths of its conformal images in the ball. We use this toshow (Theorem 2.3) that any free boundary solution has area at least π . We understandthat thisinequality is equivalent to the sharp isoperimetric inequality for free boundary surfaces. Wedefine the boundary conformal volume to be the Li-Yau conformal volume of the boundarysubmanifold.

We then proceed to define a relative conformal volume for manifolds Σ whichacceptproper conformal immersions into the unit ball. We take the maximum volume of the conformal images of a given immersion, and then minimize over conformal immersions. We show that the relative conformal volume gives a general upper bound on the first nonzero Steklov eigenvalue over all conformal metrics on Σ. Specifically we show for any k the general upper bound on $\sigma_1 \mathrm{Vol}(\partial\Sigma)(\mathrm{Vol}\,(\Sigma))^{(2-k)/2}$ in terms of the relative conformal volume. For $k = 2$ this reduces to the bound $\sigma_1 . L(\partial\Sigma) \le 2\mathrm{Vol}_{rc}(\Sigma,n)$.

II. BOUNDARY CONFORMAL VOLUME

Let (Σ^k, g) be a k-dimensional compact Riemaniann manifold with boundary $\partial\Sigma \neq \emptyset$, and let B^n be the unit ball in $\mathbb{R}^n$. supposethat Σ admits a conformal map $\varphi: \Sigma \to B^n$ with $\varphi(\partial\Sigma) \subset \partial B^n$. Let G be the group of conformal diffeomorphisms of B^n. We define the boundary conformal volume to be the Li-Yau [4] conformal volume of the boundary submanifold $\partial\Sigma$.

Definition 2.1. Given a map $\varphi \in C^1(\partial\Sigma, \partial B^n)$ that admits a conformal extension $\varphi: \Sigma \to B^n$, define the boundary n-conformal volume of φ by.

$$\mathrm{Vol}_{bc}(\Sigma,n,\varphi) = \sup_{f \in G} \mathrm{Vol}\Big(f\big(\varphi(\partial\Sigma)\big)\Big).$$

The boundary n-conformal volume of $\Sigma$is then defined to be.

$$\mathrm{Vol}_{bc}(\Sigma,n) = \inf_{\varphi} \mathrm{Vol}_{bc}(\Sigma,n,\varphi).$$

where the infimum is over all $\varphi \in C^1(\partial\Sigma, \partial B^n)$ that admit conformal extensions $\varphi: \Sigma \to B^n$.It can be shown that $\mathrm{Vol}_{bc}(\Sigma,n) \ge \mathrm{Vol}_{bc}(\Sigma,n+1)$. The boundary conformal volume of Σ is defined to be.

$$\mathrm{Vol}_{bc}(\Sigma) = \lim_{n \to \infty} \mathrm{Vol}_{bc}(\Sigma,n).$$

Note that:For any k-dimensional manifold Σ with boundary, the boundary n-conformalvolume of Σ is bounded below by the volume of the $(k-1)$-dimensional sphere:

$$\mathrm{Vol}_{bc}(\Sigma,n) \ge \mathrm{Vol}(\mathbb{S}^{k-1}).$$

The proof is as in [4]; given a point θon $\mathbb{S}^{n-1}$, let $f_\theta(t)$ be the one parameter subgroup of the group of conformal diffeomorphisms of the sphere generated by the gradient of the linear functions of $\mathbb{R}^n$ in the direction θ. For all $t, f_\theta(t)$ fixes the points $\theta$and $-\theta$, and$\lim_{t\to\infty} f_\theta(t)(x) = \theta$ for all $x \in \mathbb{S}^{n-1} \setminus \{-\theta\}$.If $\varphi: \partial\Sigma \to \mathbb{S}^{n-1}$is a map whose differential has rank $k-1$ at x, then.

$$\lim_{t\to\infty} \mathrm{Vol}\Big(f_{-\varphi(x)}(t)\big(\varphi(\partial\Sigma)\big)\Big) = m\mathrm{Vol}(\mathbb{S}^{k-1})$$

for some $m \in \mathbb{Z}^+$ (here the integer m is the multiplicity of the immersed submanifold $\partial\Sigma$ atthe point $-\theta$).

For $k = 2$ and for a minimal surface Σ that is a solution to the free boundary problem inthe unit ball B^nin $\mathbb{R}^n$, the boundary n-conformal volume of Σ is the length of the boundaryof Σ; that is, its boundary length is maximal in its conformal orbit.

Theorem 2.2. Let Σ a minimal surface in B^n, with nonempty boundary $\partial\Sigma \subset \partial B^n$, and meeting ∂B^n orthogonally along $\partial\Sigma$, given by the isometric immersion $\varphi: \Sigma \to B^n$. Then. $\mathrm{Vol}_{bc}(\Sigma, n, \varphi) = L(\partial\Sigma)$,

The length of the boundary of Σ.

Proof. The trace-free second fundamental form $\left\|A - \frac{1}{2}(Tr_g A)g\right\|^2 dV_g$ is conformally invariant for surfaces. Using the Gauss equation we have $2\left\|A - \frac{1}{2}(Tr_g A)g\right\|^2 = H^2 - 4K$. Therefore, given any $f \in G$,

$$\int_\Sigma (H^2 - 4K)da = \int_{f(\Sigma)} (\tilde{H}^2 - 4\tilde{K})d\tilde{a},$$

Where $d\tilde{a}$ denotes the induced area element on $f(\Sigma)$, and $\tilde{K}$ and$\tilde{H}$ denote the Gauss and mean curvatures of $f(\Sigma)$ in $\mathbb{R}^n$. Since Σ is minimal, $H = 0$, and so we have.

$$-4\int_\Sigma K\, da = \int_{f(\Sigma)} \tilde{H}^2\, d\tilde{a} - 4\int_{f(\Sigma)} \tilde{K}\, d\tilde{a}.(1)$$

By the Gauss-Bonnet Theorem,

$$\int_\Sigma K\, da = 2\pi\chi(\Sigma) - \int_{\partial\Sigma} k\, ds$$

$$\int_{f(\Sigma)} \tilde{K}\, da = 2\pi\chi\big(f(\Sigma)\big) - \int_{\partial f(\Sigma)} \tilde{k}\, ds,$$

and using this in (1), since $\chi(\Sigma) = \chi(f(\Sigma))$, we obtain

$$4\int_{\partial\Sigma} k\, ds = \int_{f(\Sigma)} \tilde{H}^2\, d\tilde{a} + 4\int_{\partial f(\Sigma)} \tilde{k}\, d\tilde{s}\ (2)$$
$$\geq 4\int_{\partial f(\Sigma)} \tilde{k}\, d\tilde{s}$$

If T is the oriented unit tangent vector of $\partial\Sigma$and ν is the inward unit conormal vectoralong $\partial\Sigma$, then.

$$k = \langle \frac{dT}{ds}, \nu\rangle = -\langle T, \frac{d\nu}{ds}\rangle = \langle T, \frac{d\varphi}{ds}\rangle = \langle T, T\rangle = 1,$$

where in the third to last equality we have used the fact that $\nu = -\varphi$ since Σ meets ∂B^northogonally along $\partial\Sigma$ Since f is conformal, $f(\Sigma)$ also meets ∂B^n orthogonally along $\partial f(\Sigma)$,and so we also have that $\tilde{k} = 1$. Using this in (2) we obtain.

$$L(\partial\Sigma) \geq L\big(\partial f(\Sigma)\big).$$

This shows that

$$L(\partial\Sigma) \geq \mathrm{Vol}_{bc}(\Sigma, n, \varphi)$$

as claimed.

The proof of Theorem2.2meanthat any minimal surface that is a solution to the free boundary problem in the unit ball in $\mathbb{R}^n$ has area greater than or equal to that of a flat equatorial disk solution.

Theorem 2.3 Let Σ be a minimal surface in B^n, with (nonempty) boundary $\partial\Sigma \subset \partial B^n$, and meeting ∂B^n orthogonally along $\partial\Sigma$. Then.

$$2A(\Sigma) = L(\partial\Sigma) \geq 2\pi.$$

Proof. Given $f \in G$, as in the proof Theorem 2.2, we have.

$$L(\partial\Sigma) \geq L\big(\partial f(\Sigma)\big). \qquad (3)$$

Since Σ is minimal, the coordinate functions are harmonic $\Delta_\Sigma x^i = 0$, and $\Delta_\Sigma |x|^2 = 4$. Therefore,

$$4A(\Sigma) = \int_\Sigma \Delta_\Sigma |x|^2 \, da = \int_{\partial\Sigma} \frac{\partial |x|^2}{\partial v} ds = \int_{\partial\Sigma} 2 \, ds = 2L(\partial\Sigma).$$

Using this in (3) gives.

$$2A(\Sigma) \geq L\big(\partial f(\Sigma)\big).$$

If $p \in \partial\Sigma$, then as in Remark 5.2. of [1],

$$\lim_{t\to\infty} L\left(f_p(t)(\partial\Sigma)\right) = mL(\mathbb{S}^1) = 2\pi m$$

For some $m \in \mathbb{Z}^+$, and so, we have the desired conclusion.

$$2A(\Sigma) = L(\partial\Sigma) \geq 2\pi.$$

Corollary 2.4. The sharp isoperimetric inequality holds for free boundary minimal surfacesin the ball:

$$A \leq \frac{L^2}{4\pi}.$$

Proof. For free boundary minimal surfaces in the ball we have $2A(\Sigma) = L(\partial\Sigma)$, as shown inthe proof of Theorem2.3. It follows that the inequality $A(\Sigma) \geq \pi$ is equivalent to the sharp isoperimetric inequality $\mathrm{A} \leq \mathrm{L}^2/4\pi$.

Corollary 2.5. Show that

(i) $\mathrm{Vol}_{bc}(\Sigma, n, \varphi) \leq 2\pi$
(ii) $\mathrm{Vol}_{bc}(\Sigma, n, \varphi) \leq L^2/2A$

Proof .(i) Theorem 2.2 and Theorem 2.3 shows that

$$\mathrm{Vol}_{bc}(\Sigma, n, \varphi) \leq 2\pi$$

(ii) Since $A \leq \frac{L^2}{4\pi}$ then

$$\mathrm{Vol}_{bc}(\Sigma, n, \varphi) \leq 2\pi \leq \frac{L^2}{4\pi}$$

Definition 2.6. Let Σ be a k-dimensional compact Riemannian manifold with boundary that admits a conformal map $\varphi: \Sigma \to B^n$ with $\varphi(\partial\Sigma) \subset \partial B^n$. Define the relative n-conformal volume of φ by.

$$\mathrm{Vol}_{rc}(\Sigma, n, \varphi) = \sup_{f\in G} \mathrm{Vol}\Big(\big(f(\varphi(\Sigma))\big)\Big).$$

The relative n-conformal volume of Σ is then defined to be

$$\mathrm{Vol}_{rc}(\Sigma, n) = \inf_{\varphi} \mathrm{Vol}_{rc}(\Sigma, n, \varphi)$$

Where the infimum is over all non-degenerate conformal maps

$$\varphi: \Sigma \to B^n \text{ with } \varphi(\partial\Sigma) \subset \partial B^2.$$

Lemma 2.7. If $m \geq n$, then $\mathrm{Vol}_{rc}(\Sigma, n) \geq \mathrm{Vol}_{rc}(\Sigma, m)$.

Proof. To see this, suppose $\varphi: \Sigma \to B^n \subset B^m$ is conformal, with $\varphi(\partial\Sigma) \subset \partial B^n \subset \partial B^m$. Let $A = \varphi(\Sigma) \subset B^n$ and suppose that f is a conformal transformation of B^m. Then $f(A)$ lies in the spherical cap $f(B^n)$ in B^m whose boundary lies in ∂B^m. Let $T \in O(m)$ be an orthogonal transformation that rotates this spherical cap so that its boundary lies in an n-plane parallel to the n-plane containing the boundary of the original equatorial B^n. Let P be the conformal projection of $T\big(f(B^n)\big)$onto B^n, and let $A' = P\left(T\big(f(A)\big)\right)$. Clearly P is volume increasing, and so.

$$\mathrm{Vol}(A') \geq \mathrm{Vol}\big(f(A)\big)$$

But A' is the image of A under some conformal transformation of B^n, therefore.

$$\sup_{F\in G} \mathrm{Vol}\big(F(A)\big) \geq \sup_{f\in G'} \mathrm{Vol}\big(f(A)\big)$$

Where G denotes the group of conformal transformations of B^n, and G' denotes the group of conformal transformations of B^m.

The relative conformal volume of Σ is defined to be.

$$\mathrm{Vol}_{rc}(\Sigma) = \lim_{n\to\infty} \mathrm{Vol}_{rc}(\Sigma,n)$$

Note that : For any k-dimensional manifold Σ with boundary, the relative n-conformal volume of Σ is bounded below by the volume of the k-dimensional ball:

$$\mathrm{Vol}_{rc}(\Sigma,n) \geq \mathrm{Vol}(B^k).$$

To see this, suppose $\varphi\colon \Sigma \to B^n$ is a conformal map with $\varphi(\partial\Sigma) \subset \partial B^n$, whose differential has rank k at $x \in \partial\Sigma$. The conformal diffeomorphisms $f_{-\varphi(x)}(t)$ of the sphere (see Remark5.2 of [1]), extend to conformal diffeomorphismsof B^n, and,

$$\lim_{t\to\infty} \mathrm{Vol}\Big(f_{-\varphi(x)}(t)\big(\varphi(\Sigma)\big)\Big) = m\mathrm{Vol}(B^k)$$

For some $m \in \mathbb{Z}^+$, the multiplicity of $\varphi(\partial\Sigma)$ at $\varphi(x)$.

III. FIRST EIGENVALUE

Now we prove estimates for the first eigenvalue of the Dirichlet-to-Neumann map which are analogs of the estimates of [4] and [3] for the first Neumann eigenvalue of the Laplacian, we also give Relationship between it and conformal volume (see [2]).

Corollary 3.1 We Show that $\mathrm{Vol}_{rc}(\Sigma,n) \geq \mathrm{Vol}_{rc}(\Sigma,n{+}\epsilon)$.

Proof. For $\epsilon > 0$ suppose $\varphi\colon \Sigma \to B_j^n \subset B_j^{n+\epsilon}$ is conformal, with $\varphi(\partial\Sigma) \subset B_j^n \subset \partial B_j^{n+\epsilon}$. Let $A = \varphi(\Sigma) \subset B_j^n$ and suppose that f is a conformal transformation of $B_j^{n+\epsilon}$. Then $\sum_{j=1}^r f(A_j)$ lies in the spherical cap $\sum_{j=1}^r f\big(B_j^n\big)$ in $B_j^{n+\epsilon}$. whose boundary lies in $\partial B_j^{n+\epsilon}$. Let $T \in O(m)$ be an orthogonal transformation that rotates this spherical cap so that its boundary lies in an n-plane parallel to the n-plane containing the boundary of the original equatorial B_j^n. Let P be the conformal projection of $T\big(\sum_{j=1}^r f\big(B_j^n\big)\big)$ onto B_j^n, and let $\sum_{j=1}^r A_j' = P\left(T\big(\sum_{j=1}^r f(A_j)\big)\right)$. Clearly P is volume increasing, and so.

$$\mathrm{Vol}\left(\sum_{j=1}^{r} A_j'\right) \geq \mathrm{Vol}\left(\sum_{j=1}^{r} f(A_j)\right)$$

But A_j' is the image of A_j under some conformal transformation of B_j^n, Hence.

$$\sup_{F\in G} \mathrm{Vol}\left(\sum_{j=1}^{r} f(A_j)\right) \geq \sup_{f\in G'} \mathrm{Vol}\left(\sum_{j=1}^{r} f(A_j)\right)$$

Where G is the group of conformal transformations of B_j^n, and G' denotes the group of conformal transformations of $B_j^{n+\epsilon}$.

The relative conformal volume of Σ is defined to be.

$$\mathrm{Vol}_{rc}(\Sigma) = \lim_{n\to\infty} \mathrm{Vol}_{rc}(\Sigma,n)$$

Lemma 3.2. Let (M,g) be a compact Riemanian manifold, and let φ be an immersion of M into $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. There exists $f \in G$ such that $\psi = fo\varphi = (\psi^1, \dots \psi^n)$ satisfies.

$$\int_M \psi^i \, dv_g = 0$$

For $i = 1, \dots, n$.

Proof . See [5], [3]

Theorem 3.3. Let (Σ, g) be a compact k-dimensional Riemannian manifold with nonempty boundary. Let $\sigma_1 > 0$ be the first non-zero eigenvalue of the Dirichlet-to-Neumann map on (Σ, g). Then.

$$\sigma_1 \mathrm{Vol}(\partial\Sigma)\mathrm{Vol}(\Sigma)^{\frac{2-k}{k}} \leq k\ \mathrm{Vol}_{rc}(\Sigma, n)^{\frac{2}{k}}$$

For all n for which $\mathrm{Vol}_{rc}(\Sigma, n)$ is defined (i.e. such that there exists a conformal mapping $\varphi: \Sigma \to B^n$ with $\varphi(\partial\Sigma) \subset \partial B^n$). Equality implies that there exists a conformal harmonic map $\varphi: \Sigma \to B^n$ which (after rescaling the metric g) is an isometry on $\partial\Sigma$, with $\varphi(\partial\Sigma) \subset \partial B^n$ and such that $\varphi(\Sigma)$ meets ∂B^n orthogonally along $\varphi(\partial\Sigma)$. For $k > 2$ this map is an isometric minimal immersion of Σ to its image. Moreover, the immersion is given by a subspace of the first eigenspace.

The following is an immediate consequence of the theorem.

Corollary 3.4. Let Σ be a compact surface with nonempty boundary and metric g. Let $\sigma_1 > 0$ be the first non-zero eigenvalue of the Dirichlet-to-Neumann map on (Σ, g). Then
$\sigma_1 L(\partial\Sigma) \leq 2\ \mathrm{Vol}_{rc}(\Sigma, n)$

for all n for which $\mathrm{Vol}_{rc}(\Sigma, n)$ is defined. Equality implies that there exists a conformal minimal immersion $\varphi: \Sigma \to B^n$ by first eigenfunctions which (after rescaling the metric) is anisometry on $\partial\Sigma$, with $\varphi(\partial\Sigma) \subset \partial B^n$ and such that $\varphi(\Sigma)$ meets ∂B^n orthogonally along $\varphi(\partial\Sigma)$.

Proof. Let $\varphi: \Sigma \to B^n$ be a conformal map with $\varphi(\partial\Sigma) \subset \partial B^n$. By Lemma3.2 we can assume that $\varphi = (\varphi^1, \dots, \varphi^n)$ satisfies

$$\int_{\partial\Sigma} \varphi^i\ ds = 0$$

for $i = 1, \dots, n$. Let $\hat{\varphi}^i$ be a harmonic extension of $\varphi^i|_{\partial\Sigma}$. Then,

$$\sigma_1 \leq \frac{\int_\Sigma |\nabla \hat{\varphi}^i|^2 dv_\Sigma}{\int_{\partial\Sigma} (\varphi^i)^2 dv_{\partial\Sigma}} \leq \frac{\int_\Sigma |\nabla \varphi^i|^2 dv_\Sigma}{\int_{\partial\Sigma} (\varphi^i)^2 dv_{\partial\Sigma}}. \qquad (4)$$

By Holder's inequality, and since φ is conformal

$$\int_\Sigma \sum_{i=1}^n |\nabla\varphi^i|^2 dv_\Sigma \leq \mathrm{Vol}(\Sigma)^{\frac{k-2}{k}} \left[\int_\Sigma \left(\left|\nabla\varphi^i\right|^2\right)^{\frac{k}{2}} dv_\Sigma\right]^{\frac{2}{k}} = \mathrm{Vol}(\Sigma)^{\frac{k-2}{k}} \left[k^{\frac{k}{2}} \mathrm{Vol}(\varphi(\mathrm{T}))\right]^{\frac{2}{k}}$$
$$\leq k \mathrm{Vol}(\Sigma)^{\frac{k-2}{k}} \mathrm{Vol}_{rc}(\Sigma, n, \varphi)^{\frac{2}{k}}.$$

On the other hand, since $\varphi(\partial\Sigma) \subset \partial B^n$,

$$\sum_{i=1}^n \int_{\partial\Sigma} (\varphi^i)^2 dv_{\partial\Sigma} = \int_{\partial\Sigma} dv_{\partial\Sigma} = \mathrm{Vol}(\partial\Sigma).$$

Then by (4) we have.

$$\sigma_1 \mathrm{Vol}(\partial\Sigma)\mathrm{Vol}(\Sigma)^{\frac{2-k}{k}} \leq k \mathrm{Vol}_{rc}(\Sigma, n, \varphi)^{\frac{2}{k}}.$$

Since $\mathrm{Vol}_{rc}(\Sigma, n) = \inf_\varphi \mathrm{Vol}_{rc}(\Sigma, n, \varphi)$ we get.
$\sigma_1 \mathrm{Vol}(\partial\Sigma)\mathrm{Vol}(\Sigma)^{\frac{2-k}{k}} \leq k\ \mathrm{Vol}_{rc}(\Sigma, n)^{\frac{2}{k}}.$

Now assume that we have equality, $\sigma_1 \mathrm{Vol}(\partial\Sigma) = k V_{rc}(\Sigma, n)^{2/k} V(\Sigma)^{(k-2)/k}$. Choose a sequence of conformal maps $\varphi: \Sigma \to B^n$ with $\varphi_j(\partial\Sigma) \subset \partial B^n$, such that.

$$\lim_{j\to\infty} \mathrm{Vol}_{rc}(\Sigma, n, \varphi_j) = \mathrm{Vol}_{rc}(\Sigma, n)$$

and by composing with a conformal transformation of the ball we may assume

$$\int_{\partial\Sigma} \varphi_j^i\ ds = 0$$

for all i, j. By changing the order of coordinates, we may assume that

$$\lim_{j\to\infty} \int_\Sigma (\varphi_j^i)^2\ da \begin{cases} > 0 & i = 1, \dots, N \\ = 0 & i = N+1, \dots n \end{cases}$$

We have

$$\sigma_1 \mathrm{Vol}(\partial\Sigma) = \sigma_1 \sum_{i=1}^{n} \int_{\partial\Sigma} (\varphi_j^i)^2 dv_{\partial\Sigma} \leq \sum_{i=1}^{n} \int_{\Sigma} |\nabla\varphi_j^i|^2 dv_{\Sigma} \leq \mathrm{Vol}(\Sigma)^{\frac{k-2}{k}} \left[\int_{\Sigma} \left(\sum_{i=1}^{n} |\nabla\varphi_j^i|^2 \right)^{\frac{k}{2}} dv_{\Sigma} \right]^{\frac{2}{k}}$$
$$\leq k \mathrm{Vol}_{rc}(\Sigma, n, \varphi_j)^{\frac{2}{k}} \mathrm{Vol}(\Sigma)^{\frac{k-2}{k}}$$

Letting $j \to \infty$ and using $\sigma_1 \mathrm{Vol}(\partial\Sigma) = k \mathrm{Vol}_{rc}(\Sigma, n)^{2/k} \mathrm{Vol}(\Sigma)^{(k-2)/k}$ we get

$$\sigma_1 \mathrm{Vol}(\partial\Sigma) = \sigma_1 \lim_{j\to\infty} \sum_{i=1}^{n} \int_{\partial\Sigma} (\varphi_j^i)^2 dv_{\partial\Sigma} = \lim_{j\to\infty} \sum_{i=1}^{n} \int_{\Sigma} |\nabla\varphi_j^i|^2 dv_{\Sigma} = \mathrm{Vol}(\Sigma)^{\frac{k-2}{k}} \lim_{j\to\infty} \left[\int_{\Sigma} \left(\sum_{i=1}^{n} |\nabla\varphi_j^i|^2 \right)^{\frac{k}{2}} dv_{\Sigma} \right]^{\frac{2}{k}}$$
$$= \sigma_1 \mathrm{Vol}(\partial\Sigma) (5)$$

Therefore, for any fixed $i, \{\varphi_j^i\}$ us a bounded sequence in $W^{1,k}(\Sigma, \mathbb{R})$, and since the inclusion $W^{1,k}(\Sigma, \mathbb{R}) \subset L^2(\Sigma, \mathbb{R})$ is compact, by passing to a subsequence we can assume that $\{\varphi_j^i\}$ converges weakly in $W^{1,k}(\Sigma, \mathbb{R})$, strongly in $L^2(\Sigma, \mathbb{R})$, and point wise a.e., to map $\psi^i: \Sigma \to \mathbb{R}$. Clearly $\sum_{i=1}^{n} (\psi^i)^2 \leq 1$ a.e. on $\Sigma, \sum_{i=1}^{n} (\psi^i)^2 = 1$ a.e. on $\partial\Sigma$, and $\psi^i = 0$ for $i = N+1, \dots n$. Since for all i.

$$\sigma_1 \int_{\partial\Sigma} (\varphi_j^i)^2 dv_{\partial\Sigma} \leq \int_{\Sigma} |\nabla\varphi_j^i|^2 dv_{\Sigma}.$$

And

$$\sigma_1 \lim_{j\to\infty} \sum_{i=1}^{n} \int_{\partial\Sigma} (\varphi_j^i)^2 \, dv_{\partial\Sigma} = \lim_{j\to\infty} \sum_{i=1}^{n} \int_{\Sigma} |\nabla\varphi_j^i|^2 \, dv_{\Sigma},$$

We have

$$\lim_{j\to\infty} \int_{\Sigma} |\nabla\varphi_j^i|^2 dv_{\Sigma} = \sigma_1 \lim_{j\to\infty} \int_{\partial\Sigma} (\varphi_j^i)^2 dv_{\partial\Sigma} = \sigma_1 \int_{\partial\Sigma} (\psi^i)^2 dv_{\partial\Sigma} \leq \int_{\Sigma} |\nabla\psi^i|^2 dv_{\Sigma}. \qquad (6)$$

On the other hand, $\varphi_j^i \to \psi^i$ weakly in $W^{1,k}(\Sigma, \mathbb{R})$, and so

$$\int_{\Sigma} |\nabla\psi^i|^2 \, dv_{\Sigma} \leq \lim_{j\to\infty} \int_{\Sigma} |\nabla\varphi_j^i|^2 \, dv_{\Sigma}$$

There fore, we must have equality in (6), and so

$$\lim_{j\to\infty} \int_{\Sigma} |\nabla\varphi_j^i|^2 \, dv_{\Sigma} = \int_{\Sigma} |\nabla\psi^i|^2 \, dv_{\Sigma}$$

which means $\{\varphi_j^i\}$converges to ψstrongly in $W^{1,2}(\Sigma, \mathbb{R})$. Moreover,

$$\sigma_1 \int_{\partial\Sigma} (\psi^i)^2 dv_{\partial\Sigma} = \int_{\Sigma} |\nabla\psi^i|^2 dv_{\Sigma}$$

and it follows that $\{\psi_i\}_{i=1}^{N}$ are first eigenfunctions. In particular, $\psi^i$is harmonic for $i = 1, \dots, N$. Also, since φ_j is conformal and converges strongly in $W^{1,2}$to ψ, the map

$$\psi: \ \Sigma \to B^N$$
$$x \mapsto (\psi^1(x), \dots, \psi^N(x))$$

defines a conformal map. Therefore, $\psi: \ \Sigma \to B^N$ is conformal and harmonic, with $\psi(\partial\Sigma) \subset \partial B^N$. Since $\psi(\partial\Sigma) \subset \partial B^N$and

$$\frac{\partial\psi}{\partial\nu} = \sigma_1\psi \qquad (7)$$

on$\partial\Sigma$ since $\psi^i$are eigenfunctions, it follows that $\psi(\Sigma)$ meets ∂B^Northogonally along $\psi(\partial\Sigma)$.

By scaling the metric we can assume that $\sigma_1 = 1$. Then by (7), on $\partial\Sigma$ we have

$$\left|\frac{\partial\psi}{\partial\nu}\right| = |\psi| = 1,$$

and hence $\psi$is an isometry on $\partial\Sigma$. Finally, for $k > 2$ we have from (5)

$$\lim_{j\to\infty} \sum_{i=1}^{n} \int_{\Sigma} |\nabla\varphi_j^i|^2 dv_{\Sigma} = \sum_{i=1}^{n} \int_{\Sigma} |\nabla\psi^i|^2 dv_{\Sigma} = \mathrm{Vol}(\Sigma)^{\frac{k-2}{k}} \lim_{j\to\infty} \left[\int_{\Sigma} \left(\sum_{i=1}^{n} |\nabla\varphi_j^i|^2 \right)^{\frac{k}{2}} dv_{\Sigma} \right]^{\frac{2}{k}}$$

By lower semicontinuity of the norm under weak convergence this implies

$$\int_{\Sigma} |\nabla\psi|^2 dv_{\Sigma} = \mathrm{Vol}(\Sigma)^{\frac{k-2}{k}} \left[\int_{\Sigma} \left(\sum_{i=1}^{n} |\nabla\psi^i|^2 \right)^{\frac{k}{2}} dv_{\Sigma} \right]^{\frac{2}{k}}$$

Now the Holder inequality implies the opposite inequality and thus we have equality in the Holder inequality, which implies $|\nabla\psi|^2$ is constant on Σ, and this constant must be $k$by the boundary normalization. Since $\psi$is conformal this implies that $\psi$is an isometry as claimed.

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